



# Chen's double sieve, Goldbach's conjecture and the twin prime problem

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# Chen’s double sieve, Goldbach’s conjecture and the twin prime problem

J. WU

**Abstract.** For every even integer  $N$ , denote by  $D(N)$  and  $D_{1,2}(N)$  the number of representations of  $N$  as a sum of two primes and as a sum of a prime and an integer having at most two prime factors, respectively. In this paper, we give a new upper bound for  $D(N)$  and a new lower bound for  $D_{1,2}(N)$ , which improve the corresponding results of Chen. We also obtain similar results for the twin prime problem

## Contents

§ 1. Introduction .....	1
§ 2. Preliminary lemmas .....	5
§ 3. Chen’s double sieve .....	9
§ 4. Weighted inequalities for sieve function .....	15
§ 5. Functional inequalities between $H$ and $h$ .....	22
§ 6. Proofs of Propositions 3 and 4 .....	28
§ 7. Proof of Theorem 1 .....	30
§ 8. Proof of Theorem 3 .....	32
§ 9. Chen’s system of weights .....	33
§ 10. Proofs of Theorems 2 and 5 .....	38
§ 11. Proof of Theorem 4 .....	42
References .....	47

## § 1. Introduction

Let  $\Omega(n)$  be the number of all prime factors of the integer  $n$  with the convention  $\Omega(1) = 0$ . For an even integer  $N \geq 4$ , we define  $D(N)$  as the number of representations of  $N$  as a sum of two primes:

$$D(N) := |\{p \leq N : \Omega(N - p) = 1\}|,$$

where and in what follows, the letter  $p$ , with or without subscript, denotes a prime number. The well known Goldbach conjecture can be stated as  $D(N) \geq 1$  for every even integer  $N \geq 4$ . A more precise version of this conjecture was proposed by Hardy & Littlewood [15]:

$$(1.1) \quad D(N) \sim 2\Theta(N) \quad (N \rightarrow \infty),$$

where

$$\Theta(N) := \frac{C_N N}{(\log N)^2} \quad \text{and} \quad C_N := \prod_{p|N, p>2} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

Certainly, the asymptotic formula (1.1) is extremely difficult. Although the lower bound problem remains open, the upper bound problem has a rich history. In 1949 Selberg [25] proved

$$(1.2) \quad D(N) \leq \{16 + o(1)\} \Theta(N)$$

with the help of his well known  $\lambda^2$ -upper bound sieve. By applying Linnik's large sieve method, C.D. Pan [20] in 1964 improved 16 to 12. In 1966, Bombieri & Davenport [1] obtained 8 instead. Their proof is based on the linear sieve formulas and the mean value theorem of Bombieri–Vinogradov. It seems very difficult to prove (1.2) with a constant strictly less than 8 by the method in [1]. Firstly the linear sieve formulas (see Lemma 2.2 below)

$$(1.3) \quad XV(z) f\left(\frac{\log Q}{\log z}\right) + \text{error} \leq S(\mathcal{A}; \mathcal{P}, z) \leq XV(z) F\left(\frac{\log Q}{\log z}\right) + \text{error}$$

are the best possible in the sense that taking

$$\mathcal{A} = \mathcal{B}_\nu := \{n : 1 \leq n \leq x, \Omega(n) \equiv \nu \pmod{2}\} \quad (\nu = 1, 2),$$

the upper and lower bounds in (1.3) are respectively attained by  $\nu = 1$  and  $\nu = 2$  (see [14], page 239). Secondly it is hopeless to try to improve the level of distribution  $\frac{1}{2}$  in Bombieri–Vinogradov's theorem.

In 1978, Chen [10] introduced a new idea in Selberg's sieve and proved

$$(1.4) \quad D(N) \leq 7.8342 \Theta(N) \quad (N \geq N_0).$$

His sieve machine involves two variables and is quite complicated. Roughly speaking, for the sequence

$$\mathcal{A} = \{N - p : p \leq N\}$$

he introduced two new functions  $h(s)$  and  $H(s)$  such that (1.3) holds with  $f(s) + h(s)$  and  $F(s) - H(s)$  in place of  $f(s)$  and  $F(s)$ , respectively. The key innovation is to prove  $h(s) > 0$  and  $H(s) > 0$  via three weighted inequalities (see [10], (23), (47), (64), (90), and (91)). It is worth pointing out that he did not give complete proofs for these three inequalities. Among the three inequalities, the third one is the most complicated (with 43 terms) and it seems quite difficult to reconstruct a proof. Indeed, combining any one of these three inequalities with the Chen–Iwaniec switching principle (see [7] and [16]) leads to a constant less than 8. In order to derive a better result, Chen further introduced a very complicated iterative method. In 1980, C.B. Pan [19] applied essentially the first weighted inequality of Chen to get 7.988. According to [22], Chen's proof is very long and somewhat difficult to follow, but his idea is clear.

In this paper, inspired by the ideas in [26] we shall first try to give a more comprehensive treatment on Chen's double sieve and prove an upper bound sharper than (1.4).

**Theorem 1.** *For sufficiently large  $N$ , we have*

$$D(N) \leq 7.8209 \Theta(N).$$

The improvement comes from a new weighted inequality (see Lemma 4.2 below), which is still quite complicated with 21 terms, but much simpler than Chen's third and more powerful than his second and third inequality. Recently Cai & Lu [6] give another weighted inequality (with 31 terms), which is simpler but slightly weaker than Chen's third.

One way of approaching the lower bound problem in (1.1) is to give a non trivial lower bound for the quantity

$$D_{1,2}(N) := |\{p \leq N : \Omega(N-p) \leq 2\}|.$$

In this direction, Chen [7] proved, by his system of weights and switching principle, the following famous theorem: *Every sufficiently large even integer can be written as sum of a prime and an integer having at most two prime factors.* More precisely he established

$$(1.5) \quad D_{1,2}(N) \geq 0.67 \Theta(N) \quad (N \geq N_0).$$

Then Halberstam & Richert [14] obtained a better constant 0.689 in place of 0.67 by a careful numerical calculation. As they indicated in [14], it would be interesting to know whether a more elaborate weighting procedure could be adapted to the purpose of (1.5). This might lead to numerical improvements and could be important. In 1978 Chen improved the constant 0.689 of Halberstam & Richert to 0.7544 and to 0.81 by two more elaborate systems of weights ([8], [9]). Very recently by improving Chen's weighting device Cai and Lu [5] obtained 0.8285, which they described as being near to the limit of what could be obtained by the method employed.

The second aim of this paper is to propose a larger constant.

**Theorem 2.** *For sufficiently large  $N$ , we have*

$$D_{1,2}(N) \geq 0.836 \Theta(N).$$

The proof of Theorem 2 is based on a modified version of Chen's weights (see Lemma 9.2 below), the linear sieve and the mean value theorems of Pan & Ding [21] and of Fouvry [11].

A conjecture of the same nature is the twin prime problem, which can be stated as

$$\pi_2(x) := |\{p \leq x : \Omega(p+2) = 1\}| \rightarrow \infty \quad (x \rightarrow \infty).$$

Similar to (1.1), Hardy & Littlewood [15] conjectured

$$(1.6) \quad \pi_2(x) \sim \Pi(x) \quad (x \rightarrow \infty),$$

where

$$\Pi(x) := \frac{Cx}{(\log x)^2} \quad \text{and} \quad C := 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

The methods of Selberg, Pan, Bombieri & Davenport and Chen work in a similar way and give upper bounds of this type

$$(1.7) \quad \pi_2(x) \leq \{a + o(1)\} \Pi(x),$$

where the constant  $a$  is half of the corresponding constant in the Goldbach problem. Due to the sieve of Rosser–Iwaniec and mean value theorems of Bombieri, Fouvry, Friedlander and Iwaniec, the history of (1.7) is much richer than that of (1.2). We refer the reader to [26] and [6] for a detailed historical description of this problem. In particular Wu [26] obtained 3.418 in place of  $a + o(1)$  by placing these new mean value theorems in Chen's method. The main difficulty for applying these mean value theorems in [26] is to not destroy the fact that the error terms are affected by well factorisable coefficients. Recently Cai & Lu [6] improved the constant 3.418 to 3.406. Our argument in proving Theorem 1 allows us to give a better result.

**Theorem 3.** *For sufficiently large  $x$ , we have*

$$\pi_2(x) \leq 3.3996 \Pi(x).$$

As an analogue of Theorem 2, Chen [7] proved that

$$(1.8) \quad \pi_{1,2}(x) \geq 0.335 \Pi(x) \quad (x \geq x_0),$$

where

$$\pi_{1,2}(x) := |\{p \leq x : \Omega(p+2) \leq 2\}|.$$

The constant 0.335 was improved by many mathematicians. Like (1.7), the history of (1.8) is much richer than that of (1.5). A detailed historical description on this problem can be found in the recent paper of Cai [3]. In particular he obtained 1.0974 in place of 0.335, which is an improvement of Wu's constant 1.05 [26]. Here we can propose a slightly better result.

**Theorem 4.** *For sufficiently large  $x$ , we have*

$$\pi_{1,2}(x) \geq 1.104 \Pi(x).$$

**Remark 1.** (i) Theorems 1 and 3 show that the principal terms in the linear sieve formulas can be improved in the special cases  $\mathcal{A} = \{N - p : p \leq N\}$  or  $\mathcal{A} = \{p + 2 : p \leq x\}$  (see the end of Section 3). This seems to be interesting and important. Our argument is quite general, which works for all sequences satisfying the Chen–Iwaniec switching principle.

(ii) Certainly we could obtain a better constant than 3.3996 in Theorem 3 if we used mean value theorems of ([11], Corollary 2), ([12], Lemma 6) and ([18], Proposition) as in the proof of Theorem 4. But the numerical computation involved would be quite complicated.

The Chen theorem in short intervals was first studied by Ross [23]. Let  $\alpha \in (0, 1)$  be a fixed constant and define, for  $\theta \in (0, 1)$ ,  $x \geq 2$  and even integer  $N \geq 4$ ,

$$\begin{aligned} D_{1,2}(N, \theta) &:= |\{\alpha N \leq p \leq \alpha N + N^\theta : \Omega(N - p) \leq 2\}|, \\ \pi_{1,2}(x, \theta) &:= |\{x \leq p \leq x + x^\theta : \Omega(p + 2) \leq 2\}|. \end{aligned}$$

He proved (see [28]) that for  $\theta \geq 0.98$ ,  $N \geq N_0(\theta)$  and  $x \geq x_0(\theta)$ ,

$$D_{1,2}(N, \theta) \gg \Xi(N, \theta), \quad \pi_{1,2}(x, \theta) \gg \Pi(x, \theta),$$

where

$$\Xi(N, \theta) := \frac{N^\theta}{(\log N)^2} \prod_{p|N, p>2} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right)$$

and

$$\Pi(x, \theta) := \frac{2x^\theta}{(\log x)^2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

The constant 0.98 was further improved to 0.973 by Wu [28], to 0.9729 by Salerno & Vitolo [24] and to 0.972 by Cai & Lu [4].

Our method allows us to take a smaller exponent.

**Theorem 5.** For every  $\theta \geq 0.971$ ,  $N \geq N_0(\theta)$  and  $x \geq x_0(\theta)$ , we have

$$D_{1,2}(N, \theta) \geq 0.012 \Xi(N, \theta), \quad \pi_{1,2}(x, \theta) \geq 0.006 \Pi(x, \theta).$$

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## § 2. Preliminary lemmas

This section is devoted to present the formula of the Rosser–Iwaniec linear sieve and some mean value theorem on the distribution of primes in arithmetic progressions, which will be needed later. Before stating these results, it is necessary to recall some definitions.

Let  $k$  be a positive integer and  $\tau_k(n)$  the number of ways of writing  $n$  as the product of  $k$  positive integers. An arithmetical function  $\lambda(q)$  is of *level*  $Q$  and of *order*  $k$  if

$$\lambda(q) = 0 \quad \text{for } q > Q \quad \text{and} \quad |\lambda(q)| \leq \tau_k(q) \quad \text{for } q \geq 1.$$

We say that  $\lambda$  is *well factorable* if for every decomposition  $Q = Q_1 Q_2$  ( $Q_1, Q_2 \geq 1$ ) there exist two arithmetical functions  $\lambda_1$  and  $\lambda_2$  of level  $Q_1, Q_2$  and of order  $k$  such that  $\lambda = \lambda_1 * \lambda_2$ .

**Lemma 2.1.** If  $\lambda'$  is an arithmetical function of level  $Q'$  ( $\leq Q$ ) and of order  $k'$ , then  $\lambda * \lambda'$  is well factorable of level  $QQ'$  and of order  $k + k'$ .

Let  $\mathcal{A}$  be a finite sequence of integers and  $\mathcal{P}$  a set of prime numbers. For  $z \geq 2$ , we put  $P(z) := \prod_{p < z, p \in \mathcal{P}} p$  and define the sieve function

$$S(\mathcal{A}; \mathcal{P}, z) := |\{a \in \mathcal{A} : (a, P(z)) = 1\}|.$$

If  $d$  is a square-free integer with all its prime factors belonging to  $\mathcal{P}$ , we denote by  $\mathcal{A}_d$  the set of elements of  $\mathcal{A}$  divisible by  $d$  and we write the following approximate formula

$$(2.1) \quad |\mathcal{A}_d| = \frac{w(d)}{d} X + r(\mathcal{A}, d),$$

where  $X > 1$  is independent of  $d$ , and  $w(d)$  is a multiplicative function satisfying

$$(2.2) \quad 0 \leq w(p) < p \quad \text{for } p \in \mathcal{P}.$$

We also define

$$V(z) := \prod_{p < z} \left(1 - \frac{w(p)}{p}\right)$$

and suppose that there exists an absolute constant  $K > 1$  such that

$$(2.3) \quad \frac{V(z_1)}{V(z_2)} \leq \frac{\log z_2}{\log z_1} \left(1 + \frac{K}{\log z_1}\right) \quad (z_2 \geq z_1 \geq 2).$$

The formula of the Rosser–Iwaniec linear sieve [17] is stated as follows.

**Lemma 2.2.** *Let  $0 < \varepsilon < \frac{1}{8}$  and  $2 \leq z \leq Q^{1/2}$ . Under the assumptions (2.1), (2.2) and (2.3), we have*

$$(2.4) \quad S(\mathcal{A}; \mathcal{P}, z) \leq XV(z) \left\{ F\left(\frac{\log Q}{\log z}\right) + E \right\} + \sum_{l < L} \sum_{q|P(z)} \lambda_l^+(q) r(\mathcal{A}, q)$$

and

$$(2.5) \quad S(\mathcal{A}; \mathcal{P}, z) \geq XV(z) \left\{ f\left(\frac{\log Q}{\log z}\right) + E \right\} - \sum_{l < L} \sum_{q|P(z)} \lambda_l^-(q) r(\mathcal{A}, q).$$

In these formulas,  $L$  depends only on  $\varepsilon$ , the  $\lambda_l^\pm$  are well factorable functions of order 1 and of level  $Q$ , and  $E \ll \varepsilon + \varepsilon^{-8} e^K / (\log Q)^{1/3}$ . The functions  $F, f$  are defined by

$$(2.6) \quad \begin{aligned} F(u) &= 2e^\gamma / u, & f(u) &= 0 & (0 < u \leq 2), \\ (uF(u))' &= f(u-1), & (uf(u))' &= F(u-1) & (u > 2), \end{aligned}$$

where  $\gamma$  is Euler's constant.

As usual, we denote by  $\mu(q)$  Möbius' function,  $\varphi(q)$  Euler's function and  $\nu(q)$  the number of distinct prime factors of  $q$ . Define

$$\pi(y; q, a, m) := \sum_{\substack{mp \leq y \\ mp \equiv a \pmod{q}}} 1, \quad \text{li}(y) := \int_2^y \frac{dt}{\log t}$$

and

$$\overline{E}_0(y; q, a, m) := \pi(y; q, a, m) - \frac{\text{li}(y/m)}{\varphi(q)}.$$

The next lemma is due to Pan & Ding [21], which implies Bombieri–Vinogradov's theorem. Here we state it in the form of ([22], Corollary 8.12).

**Lemma 2.3.** *Let  $f(m) \ll 1$  and  $\alpha \in (0, 1]$ . Let  $r_1(y)$  be a positive function depending on  $x$  and satisfying*

$$r_1(y) \ll x^\alpha, \quad y \leq x.$$

*Let  $r_2(m)$  be a positive function depending on  $x$  and  $y$ , and satisfying*

$$mr_2(m) \ll x, \quad m \leq x^\alpha, \quad y \leq x.$$

*Then for every  $A > 0$ , there exists a constant  $B = B(A) > 0$  such that*

$$\begin{aligned} \sum_{q \leq \sqrt{x}/(\log x)^B} \mu(q)^2 3^{\nu(q)} \max_{y \leq x} \max_{(a,q)=1} & \left| \sum_{\substack{m \leq x^{1-\alpha} \\ (m,q)=1}} f(m) \overline{E}_0(y; q, a, m) \right| \ll \frac{x}{(\log x)^A}, \\ \sum_{q \leq \sqrt{x}/(\log x)^B} \mu(q)^2 3^{\nu(q)} \max_{y \leq x} \max_{(a,q)=1} & \left| \sum_{\substack{m \leq x^{1-\alpha} \\ (m,q)=1}} f(m) \overline{E}_0(mr_1(y); q, a, m) \right| \ll \frac{x}{(\log x)^A}, \\ \sum_{q \leq \sqrt{x}/(\log x)^B} \mu(q)^2 3^{\nu(q)} \max_{y \leq x} \max_{(a,q)=1} & \left| \sum_{\substack{m \leq x^{1-\alpha} \\ (m,q)=1}} f(m) \overline{E}_0(mr_2(m); q, a, m) \right| \ll \frac{x}{(\log x)^A}. \end{aligned}$$

In order to prove Theorem 5, it is necessary to generalize the mean value theorem of Pan & Ding in short intervals. Such a result was established by Wu ([27], theorem 2).

**Lemma 2.4.** Let  $f(m) \ll 1$ ,  $\varepsilon$  be an arbitrarily small positive number and define

$$H(y, h, q, a, m) := \pi(y + h; q, a, m) - \pi(y; q, a, m) - \frac{\text{li}((y + h)/m) - \text{li}(y/m)}{\varphi(q)}.$$

Then for any  $A > 0$ , there exists a constant  $B = B(A) > 0$  such that

$$\sum_{q \leq Q} \mu(q)^2 3^{\nu(q)} \max_{(a,q)=1} \max_{h \leq x^\theta} \max_{x/2 < y \leq x} \left| \sum_{m \leq M, (m,q)=1} f(m) H(y, h, q, a, m) \right| \ll \frac{x^\theta}{(\log x)^A}$$

for  $x \geq 10$ ,  $\frac{3}{5} + \varepsilon \leq \theta \leq 1$ ,  $Q = x^{\theta-1/2}/(\log x)^B$  and  $M = x^{(5\theta-3)/2-\varepsilon}$ .

In the proofs of Theorem 3 and 4, we shall need some mean value theorems with well factorable or almost well factorable coefficients.

Let  $M \geq 1$ ,  $N \geq 1$  and  $X := MN$ . Let  $\{\alpha_m\}$  and  $\{\beta_n\}$  be two sequences of order  $k$  supported in  $[M, 2M]$  and  $[N, 2N]$  respectively. We also suppose the conditions below:

(i) For all  $B$ , the equality

$$\sum_{n \equiv n_0 \pmod{k}, (n,d)=1} \beta_n = \frac{1}{\varphi(k)} \sum_{(n,dk)=1} \beta_n + O_{B,k}(N \tau_k(d)/(\log 2N)^B)$$

holds for  $d \geq 1$ ,  $k \geq 1$  and  $(k, n_0) = 1$ .

(ii) If  $n$  has a prime factor  $p$  with  $p < \exp\{(\log \log n)^2\}$ , then  $\beta_n = 0$ .

The following result is an immediate consequence of Corollary 2 of [11], Lemma 6 of [12] and the proposition of [18].

**Lemma 2.5.** Under the conditions (i) and (ii) above, for any  $A$  and for any  $\varepsilon > 0$  we have

$$\sum_{(q,a)=1} \lambda(q) \left( \sum_{mn \equiv a \pmod{q}} \alpha_m \beta_n - \frac{1}{\varphi(q)} \sum_{(mn,q)=1} \alpha_m \beta_n \right) \ll_{\varepsilon, A} \frac{X}{(\log X)^A}$$

uniformly for  $|a| \leq (\log X)^A$  and  $\nu := \log N / \log X$  ( $\varepsilon \leq \nu \leq 1 - \varepsilon$ ). Here  $\lambda(q)$  is a well factorisable function of order 1 and of level  $Q := X^{\theta(\nu)-\varepsilon}$ , where  $\theta(\nu)$  is given by

$$\theta(\nu) = \begin{cases} \frac{6-5\nu}{10} & \text{for } \varepsilon \leq \nu \leq \frac{1}{15}, \\ \frac{1}{2} + \nu & \text{for } \frac{1}{15} \leq \nu \leq \frac{1}{10}, \\ \frac{5-2\nu}{8} & \text{for } \frac{1}{10} \leq \nu \leq \frac{3}{14}, \\ \frac{3+2\nu}{6} & \text{for } \frac{3}{14} \leq \nu \leq \frac{1}{4}, \\ \frac{2-\nu}{3} & \text{for } \frac{1}{4} \leq \nu \leq \frac{2}{7}, \\ \frac{2+\nu}{4} & \text{for } \frac{2}{7} \leq \nu \leq \frac{2}{5}, \\ 1 - \nu & \text{for } \frac{2}{5} \leq \nu \leq \frac{1}{2}, \\ \frac{1}{2} & \text{for } \frac{1}{2} \leq \nu \leq 1 - \varepsilon. \end{cases}$$

*Proof.* The value  $(6 - 5\nu)/10$  in  $[\varepsilon, 1/15]$  comes from Corollary 2 (ii) of [11]. The intervals  $[1/15, 1/10]$  and  $[1/10, 3/14]$  follow from the proposition of [18] by decomposing  $\lambda = \lambda_1 * \lambda_2$  with

$$Q_1 = Q = x^{1/2-\varepsilon}, \quad Q_2 = R = Nx^{-\varepsilon}$$

and

$$Q_1 = Q = x^{5/8-\varepsilon} N^{-5/4}, \quad Q_2 = R = Nx^{-\varepsilon},$$

respectively. The remaining case is Lemma 6 of [12].  $\square$



The next lemma is Corollary 2 (i) of [11]. This is the first result, which is valid uniformly for  $|a| \leq X$  and has the level of distribution  $> \frac{1}{2}$ .

**Lemma 2.6.** *Under the conditions (i) and (ii) above, for any  $A$  and for any  $\varepsilon > 0$  we have*

$$\sum_{(q,a)=1} \lambda(q) \left( \sum_{mn \equiv a \pmod{q}} \alpha_m \beta_n - \frac{1}{\varphi(q)} \sum_{(mn,q)=1} \alpha_m \beta_n \right) \ll_{\varepsilon,A} \frac{X}{(\log X)^A}$$

uniformly for  $|a| \leq X$  and  $\varepsilon \leq \nu := \log N / \log X \leq \frac{1}{10}$ . Here  $\lambda(q)$  is a well factorisable function of order 1 and of level  $Q := X^{5(1-\nu)9-\varepsilon}$ .

As usual define

$$\pi(y; q, a) := \sum_{p \leq y, p \equiv a \pmod{q}} 1.$$

The following result is due to Bombieri, Friedlander & Iwaniec ([2], theorem 10).

**Lemma 2.7.** *Let  $\lambda$  be a well factorable function of order  $k$  and of level  $Q = x^{4/7-\varepsilon}$ . For any  $\varepsilon > 0$  and any  $A$ , we have uniformly for  $x \geq 3$  and  $|a| \leq (\log x)^A$ ,*

$$\sum_{(q,a)=1} \lambda(q) \left( \pi(x; q, a) - \frac{\text{li}(x)}{\varphi(q)} \right) \ll_{\varepsilon,k,A} \frac{x}{(\log x)^A}.$$

When we use the weighted inequality, some coefficients are merely “almost well factorable”. So we need the following results, due to Fouvry & Grupp ([13], theorem 2 and the corollary).

**Lemma 2.8.** *Let  $\lambda$  be a well factorable function of level  $Q_1$  and of order  $k$ ,  $\xi$  an arithmetical function satisfying the conditions  $|\xi(q_2)| \leq \log x$  and  $\xi(q_2) = 0$  ( $q_2 > Q_2$ ) and let  $\Lambda$  be the von Mangoldt function. Then we have for any integer  $a$ , any  $\varepsilon > 0$  and any  $A > 0$ ,*

$$\sum_{(q_1 q_2, a)=1} \lambda(q_1) \xi(q_2) \left( \pi(x; q_1 q_2, a) - \frac{\text{li}(x)}{\varphi(q_1 q_2)} \right) \ll_{a,\varepsilon,k,A} \frac{x}{(\log x)^A},$$

so long as one of the following three conditions is true:

$$(C.1) \quad Q_2 \leq Q_1, \quad Q_1 Q_2 \leq x^{4/7-\varepsilon},$$

$$(C.2) \quad Q_2 \geq Q_1, \quad Q_1 Q_2^6 \leq x^{2-\varepsilon},$$

$$(C.3) \quad \xi(q) = \Lambda(q), \quad Q_1 Q_2 \leq x^{11/20-\varepsilon}, \quad Q_2 \leq x^{1/3-\varepsilon}.$$

The next two lemmas also are useful when we apply the switching principle.

**Lemma 2.9 ([26], Lemma 7).** *Let  $\lambda$  be a well factorable function of level  $Q := x^{4/7-\varepsilon}$  and of order  $k$ . Let  $\eta > 0$  and  $\{\varepsilon_i\}_{1 \leq i \leq r}$  be real numbers such that*

$$\varepsilon_i \geq \eta, \quad \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r = 1.$$

Then for any integer  $a$ , any  $\varepsilon > 0$  and any  $A > 0$ , we have

$$\sum_{(q,a)=1} \lambda(q) \left( \sum_{\substack{p_1 \cdots p_r \equiv a \pmod{q} \\ x^{\varepsilon_i} < p_i \leq 2x^{\varepsilon_i} \ (1 \leq i \leq r)}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(p_1 \cdots p_r, q)=1 \\ (x^{\varepsilon_i} < p_i \leq 2x^{\varepsilon_i} \ (1 \leq i \leq r)}} 1 \right) \ll_{a,\varepsilon,k,A} \frac{x}{(\log x)^A}.$$

**Lemma 2.10** ([26], Lemma 12). *Let  $x \geq 2$  and  $y = x^{1/u}$ . Then*

$$\sum_{\substack{n \leq x \\ p|n \Rightarrow p \geq y}} 1 = \frac{x}{\log y} \omega(u) + O\left(\frac{x}{(\log y)^2}\right),$$

where  $\omega(u)$  is Buchstab's function defined by

$$\omega(u) = 1/u \quad (1 \leq u \leq 2) \quad \text{and} \quad (u\omega(u))' = \omega(u-1) \quad (u \geq 2).$$

Moreover we have  $\omega(u) \leq 0.561522$  ( $u \geq 3.5$ ) and  $\omega(u) \leq 0.567144$  ( $u \geq 2$ ).

### § 3. Chen's double sieve

We shall sieve the sequence

$$\mathcal{A} := \{N - p : p \leq N\}.$$

Let  $\delta > 0$  be a sufficiently small number and  $k \in \mathbb{Z}$ . Put

$$Q := N^{1/2-\delta}, \quad \underline{d} := Q/d, \quad \mathcal{L} := \log N, \quad W_k := N^{\delta^{1+k}}.$$

Let  $\Delta$  be a real number with  $1 + \mathcal{L}^{-4} \leq \Delta < 1 + 2\mathcal{L}^{-4}$ . We put  $\mathcal{P}(N) := \{p : (p, N) = 1\}$  and denote by  $\pi_{[Y, Z]}$  the characteristic function of the set  $\mathcal{P}(N) \cap [Y, Z]$ . For  $k \in \mathbb{Z}^+$  and  $N \geq 2$ , let  $\mathfrak{U}_k(N)$  be the set of all arithmetical functions  $\sigma$  which can be written as the form

$$\sigma = \pi_{[V_1/\Delta, V_1]} * \cdots * \pi_{[V_i/\Delta, V_i]},$$

where  $i$  is an integer with  $0 \leq i \leq k$ , and  $V_1, \dots, V_i$  are real numbers satisfying

$$(3.1) \quad \begin{cases} V_1^2 \leq Q, \\ V_1 V_2^2 \leq Q, \\ \dots\dots\dots \\ V_1 \cdots V_{i-1} V_i^2 \leq Q, \\ V_1 \geq V_2 \geq \cdots \geq V_i \geq W_k. \end{cases}$$

By convention,  $\sigma$  is the characteristic function of the set  $\{1\}$  if  $i = 0$ . From this definition and Lemma 2.1, we see immediately the following result.

**Lemma 3.1.** (i) *We have  $\mathfrak{U}_k(N) \subset \mathfrak{U}_{k+1}(N)$  for  $k \in \mathbb{Z}^+$ .*

(ii) *Let  $\sigma = \pi_{[V_1/\Delta, V_1]} * \cdots * \pi_{[V_i/\Delta, V_i]} \in \mathfrak{U}_k(N)$ . Then  $\sigma$  is well factorable of level  $V := V_1 \cdots V_i$  and of order  $i$ . If  $\lambda$  is well factorable of level  $Q/V$  and of order 1, then  $\sigma * \lambda$  is well factorable of level  $Q$  and of order  $k + 1$ .*

Let  $F$  and  $f$  be defined as in (2.6) and let

$$(3.2) \quad A(s) := sF(s)/2e^\gamma \quad \text{and} \quad a(s) := sf(s)/2e^\gamma,$$

We introduce the notation

$$\Phi(N, \sigma, s) := \sum_d \sigma(d) S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s}), \quad \Theta(N, \sigma) := 4\text{li}(N) \sum_d \frac{\sigma(d) C_{dN}}{\varphi(d) \log \underline{d}}.$$

For  $k \in \mathbb{Z}^+$ ,  $N_0 \geq 2$  and  $s \in [1, 10]$ , we define  $H_{k, N_0}(s)$  and  $h_{k, N_0}(s)$  as the supremum of  $h \geq -\infty$  such that for all  $N \geq N_0$  and  $\sigma \in \mathfrak{U}_k(N)$  one has the following inequalities

$$\Phi(N, \sigma, s) \leq \{A(s) - h\} \Theta(N, \sigma)$$

and

$$\Phi(N, \sigma, s) \geq \{a(s) + h\} \Theta(N, \sigma)$$

respectively.

From this definition, we deduce immediately the following result.

**Lemma 3.2.** For  $k \in \mathbb{Z}^+$ ,  $N \geq N_0$ ,  $s \in [1, 10]$  and  $\sigma \in \mathfrak{U}_k(N)$ , we have

$$(3.3) \quad \Phi(N, \sigma, s) \leq \{A(s) - H_{k, N_0}(s)\} \Theta(N, \sigma)$$

and

$$(3.4) \quad \Phi(N, \sigma, s) \geq \{a(s) + h_{k, N_0}(s)\} \Theta(N, \sigma).$$

Obviously  $H_{k, N_0}(s), h_{k, N_0}(s)$  are decreasing on  $N_0$ , and they are also decreasing on  $k$  by Lemma 3.1. Hence we can write

$$\begin{aligned} H_k(s) &:= \lim_{N_0 \rightarrow \infty} H_{k, N_0}(s), & h_k(s) &:= \lim_{N_0 \rightarrow \infty} h_{k, N_0}(s), \\ H(s) &:= \lim_{k \rightarrow \infty} H_k(s), & h(s) &:= \lim_{k \rightarrow \infty} h_k(s). \end{aligned}$$

**Lemma 3.3.** For  $N \geq 2$  and  $\sigma = \pi_{[V_1/\Delta, V_1]} * \cdots * \pi_{[V_i/\Delta, V_i]} \in \mathfrak{U}_k(N)$ , we have

$$(3.5) \quad \mathcal{L}^{-5k} \ll_{\delta, k} \sum_d \sigma(d)/d \ll_{\delta, k} 1,$$

$$(3.6) \quad \sum_d \sigma(d) \ll_{\delta, k} V_1 \cdots V_i,$$

$$(3.7) \quad \Theta(N, \sigma) \gg_{\delta, k} N/\mathcal{L}^{5k+2}.$$

*Proof.* Let  $\sigma = \pi_{[V_1/\Delta, V_1]} * \cdots * \pi_{[V_i/\Delta, V_i]} \in \mathfrak{U}_k(N)$ . We have

$$(3.8) \quad \sum_d \frac{\sigma(d)}{d} = \prod_{1 \leq j \leq i} \sum_{p_j \in \mathcal{P}(N) \cap [V_j/\Delta, V_j]} \frac{1}{p_j}.$$

The prime number theorem of the form  $\sum_{p \leq x} 1 = \text{li}(x) + O(x e^{-2(\log x)^{1/2}})$  implies

$$\begin{aligned} \sum_{p_j \in \mathcal{P}(N) \cap [V_j/\Delta, V_j]} \frac{1}{p_j} &= \sum_{V_j/\Delta \leq p_j < V_j} \frac{1}{p_j} - \sum_{V_j/\Delta \leq p_j < V_j, p_j | N} \frac{1}{p_j} \\ &= \log \left( \frac{\log V_j}{\log(V_j/\Delta)} \right) + O \left( e^{-\log^{1/2}(V_j/\Delta)} + \frac{\mathcal{L}}{V_j \log \mathcal{L}} \right). \end{aligned}$$

Therefore our choice of  $\Delta$  and (3.1) give us

$$(3.9) \quad \sum_{p_j \in \mathcal{P}(N) \cap [V_j/\Delta, V_j]} 1/p_j \asymp_{\delta, k} \mathcal{L}^{-5}.$$

Now (3.5) follows from (3.8) and (3.9).

Since  $\sigma(d) \neq 0$  implies  $d \leq V_1 \cdots V_i$ , the second inequality in (3.5) implies (3.6). Noticing  $\Theta(N, \sigma) \gg N \mathcal{L}^{-2} \sum_d \sigma(d)/d$ , we obtain (3.7) by the first inequality in (3.5).  $\square$

**Proposition 1.** For  $k \in \mathbb{Z}^+$  and  $s \in [1, 10]$ , we have

$$H_k(s) \geq 0 \quad \text{and} \quad h_k(s) \geq 0.$$

*Proof.* We shall prove only the first inequality. The second one can be treated similarly. Let  $\sigma = \pi_{[V_1/\Delta, V_1]} * \cdots * \pi_{[V_i/\Delta, V_i]} \in \mathfrak{U}_k(N)$ . We use Lemma 2.2 with

$$X = \frac{\text{li}(N)}{\varphi(d)}, \quad w(p) = \begin{cases} p/(p-1) & \text{if } p \in \mathcal{P}(N), \\ 0 & \text{otherwise} \end{cases}$$

to estimate  $\sigma(d)S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s})$ . By Merten's formula and (3.1), we can infer that for any  $\varepsilon > 0$

$$(3.10) \quad V(\underline{d}^{1/s}) = \{1 + O_{\delta,k}(\varepsilon)\} \frac{2sC_{dN}}{e^\gamma \log \underline{d}}.$$

By using Lemma 2.2 and (3.10), we deduce

$$(3.11) \quad \sigma(d)S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s}) \leq 4\text{li}(N) \frac{\sigma(d)C_{dN}}{\varphi(d) \log \underline{d}} \left\{ A\left(\frac{\log(Q/V)}{\log \underline{d}^{1/s}}\right) + O_{\delta,k}(\varepsilon) \right\} \\ + \sum_{l < L} \sigma(d) \sum_{q|P(\underline{d}^{1/s})} \lambda_l^+(q) r(\mathcal{A}_d, q),$$

where  $\lambda_l^+(q)$  is well factorable of level  $Q/V$  with  $V := V_1 \cdots V_i$  and of order 1.

If  $\sigma(d) \neq 0$ , we have  $d \in [V/\Delta^i, V]$ , which implies  $0 \leq \log V - \log d \leq i \log \Delta \leq 2k\mathcal{L}^{-4}$ . From this we deduce that  $A(\log(Q/V)/\log \underline{d}^{1/s}) = A(s) + O_{\delta,k}(\varepsilon)$ . Inserting (3.11) and summing over  $d$ , we obtain

$$(3.12) \quad \Phi(N, \sigma, s) \leq \{A(s) + O_{\delta,k}(\varepsilon)\} \Theta(N, \sigma) + R,$$

where

$$R := \sum_{l < L} \sum_d \sigma(d) \sum_{q|P(\underline{d}^{1/s})} \lambda_l^+(q) r(\mathcal{A}_d, q).$$

Let  $q \mid P(\underline{d}^{1/s})$ . It is clear that  $\mu(q)^2 = 1$  and  $(Nd, q) = 1$ . Thus we have

$$r(\mathcal{A}_d, q) = |\mathcal{A}_{dq}| - \text{li}(N)/\varphi(dq) \\ = \pi(N; dq, N) - \text{li}(N)/\varphi(dq).$$

Hence we can see, by using Lemmas 3.1(ii) and 2.3, that

$$(3.13) \quad R \ll_\varepsilon \sum_{q \leq Q} \tau_{k+1}(q) |\pi(N; dq, N) - \text{li}(N)/\varphi(dq)| \\ \ll_{\delta,k,\varepsilon} N/\mathcal{L}^{5k+3}.$$

From (3.7), (3.12) and (3.13), we deduce

$$\Phi(N, \sigma, s) \leq \{A(s) + O_{\delta,k}(\varepsilon)\} \Theta(N, \sigma),$$

which implies, by the definition of  $H_{k,N_0}(s)$ , for any  $\varepsilon > 0$  and sufficiently large  $N_0$

$$H_{k,N_0}(s) \geq -O_{\delta,k}(\varepsilon).$$

First making  $N_0 \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$ , we obtain  $H_k(s) \geq 0$ . □

**Proposition 2.** For  $2 \leq s \leq s' \leq 10$ , we have

$$h(s) \geq h(s') + \int_{s-1}^{s'-1} \frac{H(t)}{t} dt$$

and

$$H(s) \geq H(s') + \int_{s-1}^{s'-1} \frac{h(t)}{t} dt.$$

*Proof.* We shall only prove the first inequality as the second one can be established in the same way.

Let  $k \geq 0$  and  $\sigma = \pi_{[V_1/\Delta, V_1]} * \cdots * \pi_{[V_i/\Delta, V_i]} \in \mathfrak{U}_k(N)$ . By Buchstab's identity, we write

$$(3.14) \quad \Phi(N, \sigma, s) = \Phi(N, \sigma, s') - \sum_d \sigma(d) \sum_{\underline{d}^{1/s'} \leq p < \underline{d}^{1/s}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p).$$

Next we shall give an upper bound for the last double sums  $S$ . The idea is to prove that the characteristic function of  $dp$  belongs to  $\mathfrak{U}_{k+1}(N)$ . Thus  $S$  can be estimated by a function  $H_{k+1, N_0}$ . We put  $V := V_1 \cdots V_i$ ,  $\underline{V} := Q/V$  and  $\alpha_j := \underline{V}^{1/s'} \Delta^j$ . Let  $r$  be the integer satisfying  $\alpha_r \leq \underline{V}^{1/s} < \alpha_{r+1}$ . Noticing that  $\sigma(d) \neq 0 \Rightarrow \underline{V}^{1/s'} \leq \underline{d}^{1/s'}$  and  $\underline{V}^{1/s} \leq \underline{d}^{1/s}$ , we deduce

$$(3.15) \quad \begin{aligned} S &\leq \sum_d \sigma(d) \sum_{\alpha_0 \leq p < \alpha_r} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p) + R_1 \\ &= \sum_{1 \leq j \leq r} \sum_{d, p} \sigma(d) \pi_{[\alpha_{j-1}, \alpha_j]}(p) S(\mathcal{A}_{dp}; \mathcal{P}(dpN), (\underline{dp})^{1/s^*}) + R_1, \end{aligned}$$

where  $s^* := \log \underline{d} / \log p - 1$  and

$$R_1 := \sum_d \sigma(d) \sum_{\alpha_r \leq p < \underline{d}^{1/s}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p).$$

We would prove that  $\sigma * \pi_{[\alpha_{j-1}, \alpha_j]} \in \mathfrak{U}_{k+1}(N)$ . It suffices to verify that  $V_1, V_2, \dots, V_i, \alpha_j$  satisfy (3.1) for  $j \leq r$ . If  $V_i \geq \alpha_j$ , then  $V_1 V_2 \cdots V_i \alpha_j^2 \leq V \underline{V}^{2/s} = Q^{2/s} V^{1-2/s} \leq Q$  and  $\alpha_j \geq \underline{V}^{1/s'} \geq V_i^{1/s'} \geq W_k^{1/s'} \geq W_{k+1}$ . If  $V_1 \geq \cdots \geq V_l \geq \alpha_j \geq V_{l+1} \geq \cdots \geq V_i$ , we have  $V_1 \cdots V_l \alpha_j V_{l+1} \cdots V_n^2 \leq V \alpha_j^2 \leq V \underline{V}^{2/s} \leq Q$  for  $l < n \leq i$ . Thus  $\sigma * \pi_{[\alpha_{j-1}, \alpha_j]} \in \mathfrak{U}_{k+1}(N)$ .

Since  $s^*$  depends on  $d$  and  $p$ , we replace it by a suitable quantity independent of  $d$  and  $p$  such that we can use (3.3) with  $H_{k+1, N_0}$ . For this we introduce  $s_1 := \log(\underline{V}/\alpha_j) / \log \alpha_j$ ,  $s_2 := \log(\underline{V}/\alpha_{j-1}) / \log \alpha_{j-1}$ . Noticing that  $\sigma(d) \pi_{[\alpha_{j-1}, \alpha_j]}(p) \neq 0 \Rightarrow s_1 \leq s^* \leq s_2$ , we deduce from (3.15) that

$$S \leq \sum_{1 \leq j \leq r} \sum_{d, p} \sigma(d) \pi_{[\alpha_{j-1}, \alpha_j]}(p) S(\mathcal{A}_{dp}; \mathcal{P}(dpN), (\underline{dp})^{1/s_1}) + R_1 + R_2$$

where

$$R_2 := \sum_{1 \leq j \leq r} \sum_{d, p} \sigma(d) \pi_{[\alpha_{j-1}, \alpha_j]}(p) \{S(\mathcal{A}_{dp}; \mathcal{P}(dpN), (\underline{dp})^{1/s^*}) - S(\mathcal{A}_{dp}; \mathcal{P}(dpN), (\underline{dp})^{1/s_1})\}.$$

Now we can use (3.3) in Lemma 3.2 to write

$$\begin{aligned} S &\leq \sum_{1 \leq j \leq r} \{A(s_1) - H_{k+1, N_0}(s_1)\} \Theta(N, \sigma * \pi_{[\alpha_{j-1}, \alpha_j]}) + R_1 + R_2 \\ &\leq 4\text{li}(N) \sum_d \frac{\sigma(d) C_{dN}}{\varphi(d) \log \underline{d}} \sum_{\alpha_0 \leq p < \alpha_r} \frac{A(s^*) - H_{k+1, N_0}(s^*)}{(p-2)(1 - \log p / \log \underline{d})} + R_1 + R_2 \\ &\leq 4\text{li}(N) \sum_d \frac{\sigma(d) C_{dN}}{\varphi(d) \log \underline{d}} \sum_{\underline{d}^{1/s'} \leq p < \underline{d}^{1/s}} \frac{A(s^*) - H_{k+1, N_0}(s^*)}{(p-2)(1 - \log p / \log \underline{d})} + R_1 + R_2 + R_3, \end{aligned}$$

where we have used the fact that  $A(s) - H_{k+1, N_0}(s)$  is increasing on  $s$ , and the notation

$$R_3 := 4\text{li}(N) \sum_d \frac{\sigma(d)C_{dN}}{\varphi(d) \log \underline{d}} \sum_{\underline{V}^{1/s'} \leq p < \underline{d}^{1/s'}} \frac{A(s^*) - H_{k+1, N_0}(s^*)}{\varphi(p)(1 - \log p / \log \underline{d})}.$$

Applying the prime number theorem, an integration by parts shows that

$$\sum_{\underline{d}^{1/s'} \leq p < \underline{d}^{1/s}} \frac{A(s^*) - H_{k+1, N_0}(s^*)}{(p-2)(1 - \log p / \log \underline{d})} = \int_{s-1}^{s'-1} \frac{A(t) - H_{k+1, N_0}(t)}{t} dt + O_{\delta, k}(\varepsilon).$$

Hence

$$(3.16) \quad S \leq \left\{ \int_{s-1}^{s'-1} \frac{A(t) - H_{k+1, N_0}(t)}{t} dt + O_{\delta, k}(\varepsilon) \right\} \Theta(N, \sigma) + R_1 + R_2 + R_3.$$

It remains to estimate  $R_1, R_2, R_3$ . Observing that  $\sigma(d) \neq 0 \Rightarrow V/\Delta^i \leq d < V$ , we have  $\underline{d}^{1/s} \leq \underline{V}^{1/s} \Delta^{i/s}$ . Thus  $\log(\log \underline{d}^{1/s} / \log \alpha_r) \leq \log(1 + \log \Delta^{1+i/s} / \log(\underline{V}^{1/s} / \Delta)) \ll_{\delta, k} \mathcal{L}^{-5}$ . By using the prime number theorem and the previous estimate, we have

$$(3.17) \quad \begin{aligned} R_1 &\ll \sum_d \sigma(d) \sum_{\alpha_r \leq p < \underline{d}^{1/s}} N/dp \\ &\ll N \mathcal{L}^{-5} \sum_d \sigma(d) / \varphi(d) \\ &\ll_{\delta, k} \Theta(N, \sigma) / \mathcal{L}^3. \end{aligned}$$

Similarly we can show that

$$(3.18) \quad R_3 \ll_{\delta, k} \Theta(N, \sigma) / \mathcal{L}^3.$$

By the definition of  $R_2$ , we easily see that

$$R_2 \ll \sum_d \sigma(d) \sum_{\alpha_0 \leq p < \alpha_r} \sum_{(\underline{dp})^{1/s'} \leq p' < (\underline{dp})^{1/s_1}} N/dpp'.$$

Using a similar preceding argument, we can show that

$$(3.19) \quad \begin{aligned} R_2 &\ll_{\delta, k} \frac{N}{\mathcal{L}^4} \sum_d \frac{\sigma(d)}{\varphi(d) \log \underline{d}} \sum_{\underline{V}^{1/s'} \leq p < \underline{V}^{1/s}} \frac{1}{p} \\ &\ll_{\delta, k} \frac{\Theta(N, \sigma)}{\mathcal{L}^3}. \end{aligned}$$

Combining (3.16)–(3.19), we obtain the desired upper bound, for  $N \geq N_0$ ,

$$(3.20) \quad S \leq \left\{ \int_{s-1}^{s'-1} \frac{A(t) - H_{k+1, N_0}(t)}{t} dt + O_{\delta, k}(\varepsilon) \right\} \Theta(N, \sigma).$$

Inserting it in (3.14), estimating the first sum on the right-hand side of (3.14) by (3.9) and noticing the relation

$$a(s') - a(s) = \int_s^{s'} A(t-1) dt,$$

we find that, for  $N \geq N_0(\varepsilon, \delta, k)$ ,

$$\Phi(N, \sigma, s) \geq \left\{ a(s) + h_{k, N_0}(s') + \int_{s-1}^{s'-1} \frac{H_{k+1, N_0}(t)}{t} dt + O_{\delta, k}(\varepsilon) \right\} \Theta(N, \sigma).$$

From the definition of  $h_{k, N_0}(s)$ , we deduce that, for any  $\varepsilon > 0$  and for sufficiently large  $N_0$ ,

$$h_{k, N_0}(s) \geq h_{k, N_0}(s') + \int_{s-1}^{s'-1} \frac{H_{k+1, N_0}(t)}{t} dt + O_{\delta, k}(\varepsilon).$$

Taking  $N_0 \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$ , we obtain

$$h_k(s) \geq h_k(s') + \int_{s-1}^{s'-1} \frac{H_{k+1}(t)}{t} dt$$

which implies the required inequality. This completes the proof.  $\square$

**Corollary 1.** *The function  $H(s)$  is decreasing on  $[1, 10]$ . The function  $h(s)$  is increasing on  $[1, 2]$  and is decreasing on  $[2, 10]$ .*

*Proof.* According to the definition, we easily see that  $H_{k, N_0}(s)$  is decreasing on  $[1, 3]$  since  $A(s) = 1$  for  $1 \leq s \leq 3$ . Thus  $H(s)$  is also decreasing on  $[1, 3]$ . When  $3 \leq s \leq 10$ , the required result follows immediately from Propositions 2 and 1.

Similarly the definition of  $h_{k, N_0}(s)$  and the fact that  $a(s) = 0$  for  $1 \leq s \leq 2$  show that  $h(s)$  is increasing on  $[1, 2]$ . Propositions 2 and 1 imply that  $h(s)$  is decreasing on  $[2, 10]$ . This concludes the proof.  $\square$

The central results in this section are Propositions 3 and 4 below. Before stating it, it is necessary to introduce some notation.

Let  $1 \leq s \leq 3 \leq s' \leq 5$  and  $s \leq \kappa_3 \leq \kappa_2 \leq \kappa_1 \leq s'$ . Define

$$\begin{aligned} \alpha_1 &:= \kappa_1 - 2, & \alpha_2 &:= s' - 2, & \alpha_3 &:= s' - s'/s - 1, \\ \alpha_4 &:= s' - s'/\kappa_2 - 1, & \alpha_5 &:= s' - s'/\kappa_3 - 1, & \alpha_6 &:= s' - 2s'/\kappa_2, \\ \alpha_7 &:= s' - s'/\kappa_1 - s'/\kappa_3, & \alpha_8 &:= s' - s'/\kappa_1 - s'/\kappa_2, & \alpha_9 &:= \kappa_1 - \kappa_1/\kappa_2 - 1. \end{aligned}$$

Let  $\mathbf{1}_{[a, b]}(t)$  be the characteristic function of the interval  $[a, b]$ . We put

$$\sigma(a, b, c) := \int_a^b \log \frac{c}{t-1} \frac{dt}{t}, \quad \sigma_0(t) := \frac{\sigma(3, t+2, t+1)}{1 - \sigma(3, 5, 4)}.$$

We can prove that  $H(s)$  satisfies some functional inequalities.

**Proposition 3.** *For  $5 \geq s' \geq 3 \geq s \geq 2$  and  $s' - s'/s \geq 2$ , we have*

$$(3.21) \quad H(s) \geq \Psi_1(s) + \int_1^3 H(t) \Xi_1(t; s) dt,$$

where  $\Psi_1(s)$  is defined as in Lemma 5.1 below and  $\Xi_1(t; s) = \Xi_1(t; s, s')$  is given by

$$\begin{aligned} \Xi_1(t; s) &:= \frac{\sigma_0(t)}{2t} \log \left( \frac{16}{(s-1)(s'-1)} \right) \\ &\quad + \frac{\mathbf{1}_{[\alpha_2, 3]}(t)}{2t} \log \left( \frac{(t+1)^2}{(s-1)(s'-1)} \right) \\ &\quad + \frac{\mathbf{1}_{[\alpha_3, \alpha_2]}(t)}{2t} \log \left( \frac{t+1}{(s-1)(s'-1-t)} \right). \end{aligned}$$

**Proposition 4.** Let  $5 \geq s' \geq 3 \geq s \geq 2$  and  $s \leq \kappa_3 < \kappa_2 < \kappa_1 \leq s'$  satisfy

$$s' - s'/s \geq 2, \quad 1 \leq \alpha_i \leq 3 \quad (1 \leq i \leq 9), \quad \alpha_1 < \alpha_4, \quad \alpha_5 < \alpha_8.$$

Then we have

$$(3.22) \quad H(s) \geq \Psi_2(s) + \int_1^3 H(t) \Xi_2(t; s) dt,$$

where  $\Psi_2(s)$  is defined as in Lemma 5.2 below, and  $\Xi_2(t; s) = \Xi_2(t; s, s', \kappa_1, \kappa_2, \kappa_3)$  is given by

$$\begin{aligned} \Xi_2(t; s) := & \frac{\sigma_0(t)}{5t} \log \left( \frac{1024}{(s-1)(s'-1)(\kappa_1-1)(\kappa_2-1)(\kappa_3-1)} \right) \\ & + \frac{\mathbf{1}_{[\alpha_2, 3]}(t)}{5t} \log \left( \frac{(t+1)^5}{(s-1)(s'-1)(\kappa_1-1)(\kappa_2-1)(\kappa_3-1)} \right) \\ & + \frac{\mathbf{1}_{[\alpha_9, \alpha_1]}(t)}{5t} \log \left( \frac{t+1}{(\kappa_2-1)(\kappa_1-1-t)} \right) \\ & + \frac{\mathbf{1}_{[\alpha_5, \alpha_2]}(t)}{5t} \log \left( \frac{t+1}{(\kappa_3-1)(s'-1-t)} \right) \\ & + \frac{\mathbf{1}_{[\alpha_3, \alpha_2]}(t)}{5t} \log \left( \frac{t+1}{(s-1)(s'-1-t)} \right) \\ & + \frac{\mathbf{1}_{[\alpha_1, \alpha_2]}(t)}{5t} \log \left( \frac{(t+1)^2}{(\kappa_1-1)(\kappa_2-1)} \right) \\ & + \frac{\mathbf{1}_{[\alpha_7, \alpha_5]}(t)}{5t(1-t/s')} \log \left( \frac{s'^2}{(\kappa_1 s' - s' - \kappa_1 t)(\kappa_3 s' - s' - \kappa_3 t)} \right) \\ & + \frac{\mathbf{1}_{[\alpha_5, \alpha_8]}(t)}{5t(1-t/s')} \log \left( \frac{s'(s'-1-t)}{\kappa_1 s' - s' - \kappa_1 t} \right) \\ & + \frac{\mathbf{1}_{[\alpha_6, \alpha_8]}(t)}{5t(1-t/s')} \log \left( \frac{s'}{\kappa_2 s' - s' - \kappa_2 t} \right) \\ & + \frac{\mathbf{1}_{[\alpha_8, \alpha_2]}(t)}{5t(1-t/s')} \log(s'-1-t). \end{aligned}$$

We shall prove these two propositions in Section 6. It is easy to see that  $\Xi_i(t; s)$  is positive and that for  $s \in [1, 3)$  there exist parameters  $s', \kappa_i$  such that  $\Psi_i(s) > 0$ . Therefore  $H(s) > 0$  for  $s \in [1, 3)$  and then Proposition 2 implies that  $h(s) > 0$  for  $s \in [1, 3)$ . In Sections 7 and 8, we shall give numeric solution of (3.20) and (3.21), and prove Theorems 1 and 3.

#### § 4. Weighted inequalities for sieve function

The aim of this section is to present two weighted inequalities for sieve function. The first is essentially due to Chen ([10], (23)). The second is new, which is not only much simpler than the third weighted inequality of Chen ([10], (64), (90) and (91)) but also more powerful.

**Lemma 4.1.** Let  $1 \leq s < s' \leq 10$ . For  $N \geq 2$ ,  $k \geq 0$  and  $\sigma \in \mathfrak{U}_k(N)$ , we have

$$2\Phi(N, \sigma, s) \leq \sum_d \sigma(d)(\Omega_1 - \Omega_2 + \Omega_3) + O_{\delta, k}(N^{1-\eta}),$$



where  $\eta = \eta(\delta, k) > 0$  and  $\Omega_i = \Omega_i(d)$  is given by

$$\begin{aligned}\Omega_1 &:= 2S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s'}), \\ \Omega_2 &:= \sum_{\substack{\underline{d}^{1/s'} \leq p < \underline{d}^{1/s} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), \underline{d}^{1/s'}), \\ \Omega_3 &:= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dp_1 N), p_2).\end{aligned}$$

*Proof.* By the Buchstab identity, we have

$$(4.1) \quad 2S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s}) = \Omega_1 - 2 \sum_{\substack{\underline{d}^{1/s'} \leq p < \underline{d}^{1/s} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p),$$

$$(4.2) \quad \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < \underline{d}^{1/s} \\ (p_1, N)=1}} S(\mathcal{A}_{dp_1}; \mathcal{P}(dN), p_1) = \Omega_0 + \sum_{\substack{\underline{d}^{1/s'} \leq p_1 \leq p_3 < \underline{d}^{1/s} \\ (p_1 p_3, N)=1}} S(\mathcal{A}_{dp_1 p_3}; \mathcal{P}(dp_1 N), p_3),$$

$$(4.3) \quad \sum_{\substack{\underline{d}^{1/s'} \leq p_3 < \underline{d}^{1/s} \\ (p_3, N)=1}} S(\mathcal{A}_{dp_3}; \mathcal{P}(dN), p_3) = \Omega_2 - \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_3 < \underline{d}^{1/s} \\ (p_1 p_3, N)=1}} S(\mathcal{A}_{dp_1 p_3}; \mathcal{P}(dN), p_1),$$

where

$$\Omega_0 := \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < \underline{d}^{1/s} \\ (p_1, N)=1}} S(\mathcal{A}_{dp_1}; \mathcal{P}(dN), \underline{d}^{1/s}).$$

Inserting (4.2)–(4.3) into (4.1), dropping the term  $\Omega_0$  (which is non-negative) and replacing  $p_1 \leq p_3$  by  $p_1 < p_3$ , we find that

$$2S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s}) \leq \Omega_1 - \Omega_2 + \Delta_1,$$

where

$$\begin{aligned}(4.4) \quad \Delta_1 &:= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_3 < \underline{d}^{1/s} \\ (p_1 p_3, N)=1}} \{S(\mathcal{A}_{dp_1 p_3}; \mathcal{P}(dN), p_1) - S(\mathcal{A}_{dp_1 p_3}; \mathcal{P}(dp_1 N), p_3)\} \\ &= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 \leq p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dp_1 N), p_2).\end{aligned}$$

By the inequality  $S(\mathcal{A}_{dp_1^2 p_3}; \mathcal{P}(dN), p_1) \ll N/dp_1^2 p_3$  and the fact that  $\underline{d} \geq W_k$ , we easily see

$$\sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_3 < \underline{d}^{1/s}}} S(\mathcal{A}_{dp_1^2 p_3}; \mathcal{P}(dN), p_1) \ll \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_3 < \underline{d}^{1/s}}} N/dp_1^2 p_3 \ll_{\delta, k} N^{1-\eta}/d$$

for some  $\eta = \eta(\delta, k) > 0$ . Inserting it in (4.4), we obtain that

$$(4.5) \quad \Delta_1 = \Omega_3 + O_{\delta, k}(N^{1-\eta}/d).$$

Finally we complete the proof with (3.5).  $\square$

**Lemma 4.2.** Let  $1 \leq s \leq \kappa_3 < \kappa_2 < \kappa_1 \leq s' \leq 10$ . For  $N \geq 2$ ,  $k \geq 0$  and  $\sigma \in \mathfrak{U}_k(N)$ , we have

$$5\Phi(N, \sigma, s) \leq \sum_d \sigma(d)(\Gamma_1 - \Gamma_2 - \Gamma_3 - \Gamma_4 + \Gamma_5 + \cdots + \Gamma_{21}) + O_{\delta, k}(N^{1-\eta}),$$

where  $\eta = \eta(\delta, k) > 0$  and  $\Gamma_i = \Gamma_i(d)$  is given by

$$\begin{aligned} \Gamma_1 &:= 4S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s'}) + S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/\kappa_1}), \\ \Gamma_2 &:= \sum_{\substack{\underline{d}^{1/s'} \leq p < \underline{d}^{1/s} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), \underline{d}^{1/s'}), \\ \Gamma_3 &:= \sum_{\substack{\underline{d}^{1/s'} \leq p < \underline{d}^{1/\kappa_2} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), \underline{d}^{1/s'}), \\ \Gamma_4 &:= \sum_{\substack{\underline{d}^{1/s'} \leq p < \underline{d}^{1/\kappa_3} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), \underline{d}^{1/s'}), \\ \Gamma_5 &:= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < \underline{d}^{1/\kappa_2} \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), \underline{d}^{1/s'}), \\ \Gamma_6 &:= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < \underline{d}^{1/\kappa_1}, \underline{d}^{1/\kappa_2} \leq p_2 < \underline{d}^{1/\kappa_3} \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), \underline{d}^{1/s'}), \\ \Gamma_7 &:= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < \underline{d}^{1/\kappa_1} \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_1), \\ \Gamma_8 &:= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < \underline{d}^{1/\kappa_1} \leq p_2 < \underline{d}^{1/\kappa_2} \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_1), \\ \Gamma_9 &:= \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < p_2 < p_3 < \underline{d}^{1/\kappa_3} \\ (p_1 p_2 p_3, N)=1}} \sum \sum S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2), \\ \Gamma_{10} &:= \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < p_2 < \underline{d}^{1/\kappa_2} \leq p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum \sum S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2), \\ \Gamma_{11} &:= \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_2} \leq p_2 < p_3 < \underline{d}^{1/\kappa_3} \\ (p_1 p_2 p_3, N)=1}} \sum \sum S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2), \\ \Gamma_{12} &:= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < \underline{d}^{1/\kappa_1}, \underline{d}^{1/\kappa_3} \leq p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum \sum S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2), \\ \Gamma_{13} &:= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < \underline{d}^{1/\kappa_1} \leq p_2 < \underline{d}^{1/\kappa_2} \leq p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum \sum S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2), \\ \Gamma_{14} &:= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < \underline{d}^{1/\kappa_1}, \underline{d}^{1/\kappa_2} \leq p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum \sum S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2), \end{aligned}$$

$$\begin{aligned}
\Gamma_{15} &:= \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_2} \leq p_2 < \underline{d}^{1/\kappa_3} \leq p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2), \\
\Gamma_{16} &:= \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < p_2 < p_3 < p_4 < \underline{d}^{1/\kappa_3} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4}; \mathcal{P}(dN), p_3), \\
\Gamma_{17} &:= \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < p_2 < p_3 < \underline{d}^{1/\kappa_3} \leq p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4}; \mathcal{P}(dN), p_3), \\
\Gamma_{18} &:= \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < p_2 < \underline{d}^{1/\kappa_3} \leq p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4}; \mathcal{P}(dN), p_3), \\
\Gamma_{19} &:= \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_2}, \underline{d}^{1/\kappa_3} \leq p_2 < p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4}; \mathcal{P}(dN), p_3), \\
\Gamma_{20} &:= \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < \underline{d}^{1/\kappa_3} \leq p_2 < p_3 < p_4 < p_5 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4 p_5, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4 p_5}; \mathcal{P}(dN), p_4), \\
\Gamma_{21} &:= \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p_1 < p_2 < p_3 < p_4 < p_5 < p_6 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4 p_5 p_6, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4 p_5 p_6}; \mathcal{P}(dN), p_5).
\end{aligned}$$

*Proof.* Let  $S := S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s})$ . By using the Buchstab identity, we have

$$\begin{aligned}
(4.6) \quad 2S &= 2S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s'}) - \sum_{\substack{\underline{d}^{1/s'} \leq p < \underline{d}^{1/s} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p) \\
&\quad - \Gamma_3 + \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < \underline{d}^{1/\kappa_2} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_1) - \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p < \underline{d}^{1/s} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p) \\
&=: 2S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s'}) - E_1 - \Gamma_3 + D'_1 - E_2.
\end{aligned}$$

We can also write, always by Buchstab's identity,

$$\begin{aligned}
(4.7) \quad S &= S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s'}) - \sum_{\substack{\underline{d}^{1/s'} \leq p < \underline{d}^{1/\kappa_3} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p) \\
&\quad - \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p < \underline{d}^{1/s} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p).
\end{aligned}$$

But we have

$$\begin{aligned}
\sum_{\substack{\underline{d}^{1/s'} \leq p < \underline{d}^{1/\kappa_3} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p) &= \Gamma_4 - \Gamma_7 - \Gamma_8 - \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < p_2 < \underline{d}^{1/\kappa_3} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_1) \\
&\quad - \Gamma_6 + \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < \underline{d}^{1/\kappa_1}, \underline{d}^{1/\kappa_2} \leq p_3 < \underline{d}^{1/\kappa_3} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_1).
\end{aligned}$$

Inserting these relations into (4.7), it yields that

$$\begin{aligned}
(4.8) \quad S &= S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s'}) - \Gamma_4 + \Gamma_6 + \Gamma_7 + \Gamma_8 \\
&\quad - \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p < \underline{d}^{1/s} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p) + \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < p_2 < \underline{d}^{1/\kappa_3} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_1) \\
&\quad - \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < \underline{d}^{1/\kappa_1}, \underline{d}^{1/\kappa_2} \leq p_3 < \underline{d}^{1/\kappa_3} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_1) \\
&=: S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s'}) - \Gamma_4 + \Gamma_6 + \Gamma_7 + \Gamma_8 - E_3 + D_2 - E_4.
\end{aligned}$$

Similar to (4.8), we can prove that

$$\begin{aligned}
(4.9) \quad S &= S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s'}) - \Gamma_2 + \Gamma_5 - \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < p_3 < \underline{d}^{1/\kappa_2} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_1) \\
&\quad + \left\{ \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < \underline{d}^{1/\kappa_2} \leq p_2 < \underline{d}^{1/s} \\ (p_1 p_2, N)=1}} + \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < p_2 < \underline{d}^{1/s} \\ (p_1 p_2, N)=1}} \right\} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_1) \\
&=: S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s'}) - \Gamma_2 + \Gamma_5 - E_5 + D_1'' + D_1'''.
\end{aligned}$$

Finally we write

$$(4.10) \quad S = S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/\kappa_1}) - \left\{ \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p < \underline{d}^{1/\kappa_3} \\ (p, N)=1}} + \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p < \underline{d}^{1/s} \\ (p, N)=1}} \right\} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p).$$

For  $p_1 < \underline{d}^{1/\kappa_3} < \underline{d}^{1/s}$ , we have

$$S(\mathcal{A}_{dp}; \mathcal{P}(dN), p) \geq \sum_{\substack{p \leq p_1 < \underline{d}^{1/\kappa_3} \\ (p_1, N)=1}} S(\mathcal{A}_{dp p_1}; \mathcal{P}(dN), p_1) + \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p_1 < \underline{d}^{1/s} \\ (p_1, N)=1}} S(\mathcal{A}_{dp p_1}; \mathcal{P}(dN), p_1).$$

This implies

$$\begin{aligned}
\sum_{\substack{\underline{d}^{1/\kappa_1} \leq p < \underline{d}^{1/\kappa_3} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p) &\geq \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < p_2 < \underline{d}^{1/\kappa_3} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_2) \\
&\quad + \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_3} \leq p_2 < \underline{d}^{1/s} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_2).
\end{aligned}$$

Inserting it into (4.10), we obtain

$$\begin{aligned}
(4.11) \quad S &\leq S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/\kappa_1}) - \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < p_2 < \underline{d}^{1/\kappa_3} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_2) \\
&\quad - \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_3} \leq p_2 < \underline{d}^{1/s} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_2) - \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p < \underline{d}^{1/s} \\ (p, N)=1}} S(\mathcal{A}_{dp}; \mathcal{P}(dN), p) \\
&=: S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/\kappa_1}) - E_6 - E_7 - E_3.
\end{aligned}$$

Now by adding up the inequalities (4.6), (4.8), (4.9) and (4.11) and by noticing the estimate  $D_2 - E_6 \leq \Gamma_9 + O_{\delta,k}(N^{1-\eta}/d)$ , we get

$$(4.12) \quad 5S \leq \Gamma_1 - \Gamma_2 - \Gamma_3 - \Gamma_4 + \Gamma_5 + \cdots + \Gamma_9 + \Delta_2 + O_{\delta,k}(N^{1-\eta}/d)$$

where

$$\Delta_2 := D_1 - E_1 - E_2 - 2E_3 - E_4 - E_5 - E_7$$

and

$$D_1 := D'_1 + D''_1 + D'''_1 = \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < \underline{d}^{1/s} \\ (p_1 p_2, N)=1}} \sum_{p_1 < p_2} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_1).$$

Clearly we have

$$E_1 \geq \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < \underline{d}^{1/s} \\ (p_1 p_2, N)=1}} \sum_{p_1 < p_2} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_2).$$

Thus an application of Bechstab's identity gives us

$$D_1 - E_1 \leq \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum_{p_1 < p_2} \sum_{p_2 < p_3} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2) + O_{\delta,k}(N^{1-\eta}/d).$$

From this, we can deduce

$$\begin{aligned} D_1 - E_1 - E_5 &\leq \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum_{p_1 < p_2} \sum_{p_2 < p_3} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2) \\ &\quad - \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < p_3 < \underline{d}^{1/\kappa_2} \\ (p_1 p_2 p_3, N)=1}} \sum_{p_1 < p_2} \sum_{p_2 < p_3} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2) + O_{\delta,k}(N^{1-\eta}/d) \\ &= \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < p_2 < \underline{d}^{1/\kappa_2} \leq p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum_{p_1 < p_2} \sum_{p_2 < p_3} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2) \\ &\quad + \sum_{\substack{\underline{d}^{1/s'} \leq p_1 < \underline{d}^{1/\kappa_2} \leq p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum_{p_1 < p_2} \sum_{p_2 < p_3} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2) \\ &\quad + \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum_{p_1 < p_2} \sum_{p_2 < p_3} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2) + O_{\delta,k}(N^{1-\eta}/d) \\ &=: D_3 + D_4 + D_5 + O_{\delta,k}(N^{1-\eta}/d). \end{aligned}$$

We have

$$D_3 - E_4 \leq \Gamma_{10} + \Gamma_{12} + \Gamma_{13}.$$

By splitting  $D_4$  into 4 subsums, we have

$$D_4 = \Gamma_{11} + \Gamma_{14} + \Gamma_{15} + \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_2}, \underline{d}^{1/\kappa_3} \leq p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum_{p_1 < p_2} \sum_{p_2 < p_3} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2).$$

Similarly by splitting  $E_7$  into 2 subsums, we have

$$\begin{aligned} E_7 &= E_8 + \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_2}, \underline{d}^{1/\kappa_3} \leq p_2 < \underline{d}^{1/s} \\ (p_1 p_2, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_2}, \underline{d}^{1/\kappa_3} \leq p_2 < \underline{d}^{1/s} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_2) \\ &\geq E_8 + \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_2}, \underline{d}^{1/\kappa_3} \leq p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_2}, \underline{d}^{1/\kappa_3} \leq p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_3), \end{aligned}$$

where

$$E_8 := \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < \underline{d}^{1/\kappa_3} \leq p_2 < \underline{d}^{1/s} \\ (p_1 p_2, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < \underline{d}^{1/\kappa_3} \leq p_2 < \underline{d}^{1/s} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{dp_1 p_2}; \mathcal{P}(dN), p_2).$$

By noticing that

$$\begin{aligned} &\sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_2}, \underline{d}^{1/\kappa_3} \leq p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_1} \leq p_1 < \underline{d}^{1/\kappa_2}, \underline{d}^{1/\kappa_3} \leq p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \{S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_2) - S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_3)\} \\ &= \Gamma_{19} + O_{\delta, k}(N^{1-\eta}/d), \end{aligned}$$

we can deduce

$$D_4 - E_7 \leq \Gamma_{11} + \Gamma_{14} + \Gamma_{15} + \Gamma_{19} - E_8 + O_{\delta, k}(N^{1-\eta}/d).$$

Since

$$E_2 \geq \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < p_2 < p_3 < \underline{d}^{1/s} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3}; \mathcal{P}(dN), p_3),$$

we have

$$\begin{aligned} D_5 - E_2 &\leq \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < p_2 < p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < p_2 < p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4}; \mathcal{P}(dN), p_3) + O_{\delta, k}(N^{1-\eta}/d) \\ &=: D_6 + O_{\delta, k}(N^{1-\eta}/d). \end{aligned}$$

Similarly

$$\begin{aligned} E_3 &\geq \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p_1 \leq p_2 < p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p_1 \leq p_2 < p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4}; \mathcal{P}(dN), p_4) =: E'_3, \\ E_3 &\geq \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p_1 \leq p_2 < p_3 < p_4 < p_5 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4 p_5, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p_1 \leq p_2 < p_3 < p_4 < p_5 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4 p_5, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4 p_5}; \mathcal{P}(dN), p_5) =: E''_3, \\ E_8 &\geq \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < \underline{d}^{1/\kappa_3} \leq p_2 \leq p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < \underline{d}^{1/\kappa_3} \leq p_2 \leq p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4}; \mathcal{P}(dN), p_4) =: E'_8, \\ D_6 &= \Gamma_{16} + \Gamma_{17} + \Gamma_{18} + \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < \underline{d}^{1/\kappa_3} \leq p_2 < p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_2} \leq p_1 < \underline{d}^{1/\kappa_3} \leq p_2 < p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4}; \mathcal{P}(dN), p_3) \\ &\quad + \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p_1 < p_2 < p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p_1 < p_2 < p_3 < p_4 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{dp_1 p_2 p_3 p_4}; \mathcal{P}(dN), p_3) \\ &=: \Gamma_{16} + \Gamma_{17} + \Gamma_{18} + D'_6 + D''_6. \end{aligned}$$

Since

$$D'_6 - E'_8 = \Gamma_{20} + O_{\delta,k}(N^{1-\eta}/d)$$

and

$$\begin{aligned} D''_6 - E'_3 - E''_3 &= \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p_1 < p_2 < p_3 < p_4 < p_5 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4 p_5, N)=1}} \sum \sum \sum \sum \sum S(\mathcal{A}_{dp_1 p_2 p_3 p_4 p_5}; \mathcal{P}(dN), p_4) \\ &\quad - \sum_{\substack{\underline{d}^{1/\kappa_3} \leq p_1 < p_2 < p_3 < p_4 < p_5 < \underline{d}^{1/s} \\ (p_1 p_2 p_3 p_4 p_5, N)=1}} \sum \sum \sum \sum \sum S(\mathcal{A}_{dp_1 p_2 p_3 p_4 p_5}; \mathcal{P}(dN), p_5) + O_{\delta,k}(N^{1-\eta}/d) \\ &= \Gamma_{21} + O_{\delta,k}(N^{1-\eta}/d), \end{aligned}$$

we have

$$\begin{aligned} D_6 - 2E_3 - E_8 &\leq \Gamma_{16} + \Gamma_{17} + \Gamma_{18} + D'_6 + D''_6 - E'_3 - E''_3 - E'_8 \\ &= \Gamma_{16} + \Gamma_{17} + \Gamma_{18} + \Gamma_{20} + \Gamma_{21} + O_{\delta,k}(N^{1-\eta}/d). \end{aligned}$$

Combining these estimations leads to the following inequalities

$$\begin{aligned} (4.13) \quad \Delta_2 &\leq D_3 + D_4 + D_5 - E_2 - 2E_3 - E_4 - E_7 \\ &\leq \Gamma_{10} + \cdots + \Gamma_{15} + \Gamma_{19} + D_6 - 2E_3 - E_8 + O_{\delta,k}(N^{1-\eta}/d) \\ &\leq \Gamma_{10} + \cdots + \Gamma_{21} + O_{\delta,k}(N^{1-\eta}/d). \end{aligned}$$

Now the desired result follows from (4.12) and (4.13).  $\square$

### § 5. Functional inequalities between $H(s)$ and $h(s)$

In this section, we start from two weighted inequalities for the sieve function to deduce two functional inequalities between  $H(s)$  and  $h(s)$ . They will be used to prove Propositions 3 and 4 in the next section.

**Lemma 5.1.** *For  $5 \geq s' \geq 3 \geq s \geq 2$  and  $s' - s'/s \geq 2$ , we have*

$$H(s) \geq \Psi_1(s) + \frac{1}{2} \int_{1-1/s}^{1-1/s'} \frac{h(s't)}{t(1-t)} dt + H(s'),$$

where  $\Psi_1(s) = \Psi_1(s, s')$  is given by

$$\Psi_1(s) := - \int_2^{s'-1} \frac{\log(t-1)}{t} dt + \frac{1}{2} \int_{1-1/s}^{1-1/s'} \frac{\log(s't-1)}{t(1-t)} dt - I_1(s)$$

and  $I_1(s) = I_1(s, s')$  is given by

$$I_1(s) := \max_{\phi \geq 2} \int \int \int_{1/s' \leq t \leq u \leq v \leq 1/s} \omega\left(\frac{\phi - t - u - v}{u}\right) \frac{dt du dv}{tu^2 v}.$$

*Proof.* Our starting point is the inequality in Lemma 4.1. We need to estimate all terms in the right-hand side of this inequality.

Firstly, (3.3) of Lemma 3.2 gives us

$$(5.1) \quad \sum_d \sigma(d) \Omega_1 \leq 2\{A(s') - H_{k, N_0}(s')\} \Theta(N, \sigma).$$

Secondly, by an argument similar to the proof of (3.20), we can prove, for any  $\varepsilon > 0$  and  $N \geq N_0(\varepsilon, \delta, k)$ ,

$$(5.2) \quad \sum_d \sigma(d) \Omega_2 \geq \left\{ \int_{1-1/s}^{1-1/s'} \frac{a(s't) + h_{k+1, N_0}(s't)}{t(1-t)} dt - \varepsilon \right\} \Theta(N, \sigma).$$

Finally we apply the switching principle to estimate  $\sum_d \sigma(d) \Omega_3$ . For this, we introduce

$$\mathcal{E} := \{e : e = dnp_1p_2, \sigma(d) \neq 0, (n, p_1, p_2) \text{ satisfies (5.3) below}\},$$

where

$$(5.3) \quad \underline{d}^{1/s'} \leq p_1 < p_2 < \underline{d}^{1/s}, \quad (p_1p_2, dN) = 1, \quad n \leq N/dp_1p_2^2, \quad (n, NP(p_2)/p_1) = 1;$$

and

$$\mathcal{B} := \{b : b = N - ep_3, e \in \mathcal{E}, p_2 < p_3 \leq \kappa(d, e)\},$$

where  $\kappa(d, e) := \min\{N/e, \underline{d}^{1/s}\}$ . The set  $\mathcal{B}$  is a multiset and an element  $b$  may occur more than once.

Clearly  $\sum_d \sigma(d) \Omega_3$  does not exceed the number of primes in the set  $\mathcal{B}$ . Thus

$$\sum_d \sigma(d) \Omega_3 \leq S(\mathcal{B}; \mathcal{P}(N), Q^{1/2}) + O(Q^{1/2}).$$

In the set  $\mathcal{E}$ ,  $d$  is not determined uniquely by  $e$ . This causes technical difficulty. In order to avoid it, we define  $\mathcal{E}'$  and  $\mathcal{B}'$ , similar to  $\mathcal{E}$  and  $\mathcal{B}$ , with the condition  $(n, NP(p_2)/p_1) = 1$  replaced by  $(n, dNP(p_2)/p_1) = 1$  and  $\mathcal{E}$  by  $\mathcal{E}'$  respectively. Obviously the difference  $S(\mathcal{B}; \mathcal{P}(N), Q^{1/2}) - S(\mathcal{B}'; \mathcal{P}(N), Q^{1/2})$  is

$$\leq \sum_d \sigma(d) \sum_{\underline{d}^{1/s'} \leq p_1 < p_2 < p_3 < \underline{d}^{1/s}} \sum_{\substack{n \leq N/dp_1p_2p_3 \\ (n, d)=1}} 1 \ll_{\delta, k} N^{1-\eta}$$

where  $\eta = \eta(\delta, k) > 0$ . Hence

$$(5.4) \quad \sum_d \sigma(d) \Omega_3 \leq S(\mathcal{B}'; \mathcal{P}(N), Q^{1/2}) + O_{\delta, k}(N^{1-\eta}).$$

In order to estimate  $S(\mathcal{B}'; \mathcal{P}(N), Q^{1/2})$ , we use Theorem 5.2 in [14] with

$$X := \sum_{e \in \mathcal{E}'} \sum_{p_2 < p_3 \leq \kappa(d, e)} 1, \quad w(q) = \begin{cases} q/\varphi(q) & \text{if } \mu(q)^2 = (q, N) = 1 \\ 0 & \text{otherwise} \end{cases}$$

to write

$$(5.5) \quad S(\mathcal{B}; \mathcal{P}(N), Q^{1/2}) \leq 8(1 + \varepsilon) \frac{C_N X}{\log N} + O(R_1 + R_2),$$

where

$$\begin{aligned} R_1 &:= \sum_{\substack{q < Q \\ q|P(Q^{1/2})}} 3^{\nu_1(q)} \left| \sum_{\substack{e \in \mathcal{E}' \\ (e, q)=1}} \left( \sum_{\substack{p_2 < p_3 \leq \kappa(d, e) \\ ep_3 \equiv N \pmod{q}}} 1 - \frac{1}{\varphi(q)} \sum_{p_2 < p_3 \leq \kappa(d, e)} 1 \right) \right|, \\ R_2 &:= \sum_{\substack{q < Q \\ q|P(Q^{1/2})}} 3^{\nu_1(q)} / \varphi(q) \sum_{\substack{e \in \mathcal{E}' \\ (e, q) > 1}} \sum_{p_2 < p_3 \leq \kappa(d, e)} 1. \end{aligned}$$



We first estimate  $R_2$ . Noticing that for  $e \in \mathcal{E}'$  we have  $e \leq N^{1-\eta}$  and the smallest prime factor of  $e$  is  $\geq \min\{p_1, V_k/\Delta\} \geq W_k^{1/s'}$ , we can deduce

$$\begin{aligned}
 R_2 &\leq N \sum_{q \leq Q} \frac{3^{\nu_1(q)}}{\varphi(q)} \sum_{\substack{e \leq N^{1-\eta} \\ (e,q) \geq W_k^{1/s'}}} \frac{1}{e} \\
 &\ll N \mathcal{L} \sum_{m \geq W_k^{1/s'}} \frac{1}{m} \sum_{\substack{q \leq Q \\ q \equiv 0 \pmod{m}}} \frac{3^{\nu_1(q)}}{\varphi(q)} \\
 (5.6) \quad &\ll N W_k^{1/3s'} \sum_{m \geq W_k^{1/s'}} \frac{1}{m} \sum_{\substack{q \leq Q \\ q \equiv 0 \pmod{m}}} \frac{1}{q} \\
 &\ll N W_k^{1/2s'} \sum_{m \geq W_k^{1/s'}} \frac{1}{m^2} \\
 &\ll_{\delta,k} \frac{\Theta(N, \sigma)}{\mathcal{L}^3}.
 \end{aligned}$$

Next we estimate  $R_1$ . Let  $g(a) := \sum_{e \in \mathcal{E}', e=a} 1$ . Obviously for each  $e = dnp_1p_2 \in \mathcal{E}'$ , the integers  $d, n, p_1, p_2$  are pairwise coprime. Therefore they are uniquely determined by  $e$ . Thus  $g(a) \leq 1$  and there are some injections  $r_0(e) = d$  and  $r(e) = p_2$ . Then we have

$$R_1 \ll \sum_{\substack{q \leq Q \\ (q,N)=1}} \mu(q)^2 3^{\nu_1(q)} \left| \sum_{\substack{a \in I(a) \\ (a,q)=1}} g(a) \left( \sum_{\substack{p_2 < p_3 \leq \kappa(d,a) \\ ap_3 \equiv N \pmod{q}}} 1 - \frac{1}{\varphi(q)} \sum_{p_2 < p_3 \leq \kappa(d,a)} 1 \right) \right|,$$

where  $I(a) := (r_0(a) \underline{r_0(a)}^{2/s'}, N / \underline{r_0(a)}^{1/s'})$ .

Since  $\underline{d}^{1/s'} < r(e) < \underline{d}^{1/s}$  and  $er(e) \leq N$ , we can write

$$R_1 \ll R_1^{(1)} + R_1^{(2)} + R_1^{(3)},$$

where

$$\begin{aligned}
 R_1^{(1)} &:= \sum_{\substack{q \leq Q \\ (q,N)=1}} \mu(q)^2 3^{\nu_1(q)} \left| \sum_{\substack{a \in I_1(a) \\ (a,q)=1}} g(a) \left( \sum_{\substack{p_3 \leq \underline{r_0(a)}^{1/s} \\ ap_3 \equiv N \pmod{q}}} 1 - \frac{1}{\varphi(q)} \sum_{p_3 \leq \underline{r_0(a)}^{1/s}} 1 \right) \right|, \\
 R_1^{(2)} &:= \sum_{\substack{q \leq Q \\ (q,N)=1}} \mu(q)^2 3^{\nu_1(q)} \left| \sum_{\substack{a \in I_2(a) \\ (a,q)=1}} g(a) \left( \sum_{\substack{p_3 \leq N/a \\ ap_3 \equiv N \pmod{q}}} 1 - \frac{1}{\varphi(q)} \sum_{p_3 \leq N/a} 1 \right) \right|, \\
 R_1^{(3)} &:= \sum_{\substack{q \leq Q \\ (q,N)=1}} \mu(q)^2 3^{\nu_1(q)} \left| \sum_{\substack{a \in I(a) \\ (a,q)=1}} g(a) \left( \sum_{\substack{p_3 \leq r(a) \\ ap_3 \equiv N \pmod{q}}} 1 - \frac{1}{\varphi(q)} \sum_{p_3 \leq r(a)} 1 \right) \right|,
 \end{aligned}$$

and  $I_1(a) := (r_0(a) \underline{r_0(a)}^{2/s'}, N / \underline{r_0(a)}^{1/s}]$ ,  $I_2(a) := (N / \underline{r_0(a)}^{1/s}, N / \underline{r_0(a)}^{1/s'})$ .

Applying Lemma 2.3 yields  $R_1^{(j)} \ll_{\delta,k} N / \mathcal{L}^{5k+5}$  for  $j = 1, 2, 3$ . Hence

$$(5.7) \quad R_1 \ll \Theta(N, \sigma) / \mathcal{L}^3.$$

Replacing  $(n, NP(p_2)/p_1) = 1$  by  $(n, NP(p_2)) = 1$  in the definition of  $X$ , the difference is  $\ll_{\delta,k} N\mathcal{L}^2/d\mathcal{L}^{1/s'} \ll_{\delta,k} N/\mathcal{L}^2d$ . Thus we can obtain, by Lemma 2.10, that

$$\begin{aligned} X &= \sum_d \sigma(d) \left\{ \sum_{\substack{\mathcal{L}^{1/s'} \leq p_1 < p_2 < p_3 < \mathcal{L}^{1/s} \\ (p_1 p_2, dN)=1}} \sum_{\substack{n \leq N/dp_1 p_2 p_3 \\ (n, NP(p_2))=1}} 1 + O_{\delta,k}(N/\mathcal{L}^2d) \right\} \\ &\leq (1+\varepsilon) \sum_d \sigma(d) \left\{ \sum_{\mathcal{L}^{1/s'} \leq p_1 < p_2 < p_3 < \mathcal{L}^{1/s}} \frac{N\omega(\log(N/dp_1 p_2 p_3)/\log p_2)}{dp_1 p_2 p_3 \log p_2} + O_{\delta,k}\left(\frac{N}{\mathcal{L}^2d}\right) \right\}. \end{aligned}$$

By applying the prime number theorem, we can deduce

$$X \leq (1+\varepsilon)N \sum_d \frac{\sigma(d)}{d \log \mathcal{L}} \int \int \int_{1/s' \leq t \leq u \leq v \leq 1/s} \omega\left(\frac{\phi_{d,N} - t - u - v}{u}\right) \frac{dt du dv}{tu^2 v},$$

where  $\phi_{d,N} := \log(N/d)/\log \mathcal{L}$ . Obviously  $\sigma(d) \neq 0$  implies  $\phi_{d,N} \geq 2$ . Thus

$$(5.8) \quad X \leq (1+\varepsilon)I_1(s)N \sum_d \frac{\sigma(d)}{d \log \mathcal{L}} \leq (1+\varepsilon)I_1(s)N \sum_d \frac{\sigma(d)}{\varphi(d) \log \mathcal{L}}.$$

Combining (5.4)–(5.8) and noticing  $C_N \leq C_{dN}$ , we obtain, for any  $\varepsilon > 0$  and  $N \geq N_0(\varepsilon, \delta, k)$ ,

$$(5.9) \quad \sum_d \sigma(d)\Omega_3 \leq \{2I_1(s) + \varepsilon\}\Theta(N, \sigma).$$

Inserting (5.1), (5.2) and (5.9) into the inequality of Lemma 4.1 and noticing that

$$(5.10) \quad A(s') = A(s) + \int_2^{s'-1} \frac{\log(t-1)}{t} dt, \quad a(s't) = \log(s't-1),$$

we find that

$$\Phi(N, \sigma, s) \leq \left\{ A(s) - \Psi_1(s) - \frac{1}{2} \int_{1-1/s}^{1-1/s'} \frac{h_{k+1, N_0}(s't)}{t(1-t)} dt - H_{k, N_0}(s') + \varepsilon \right\} \Theta(N, \sigma).$$

By the definition of  $H_{k, N_0}(s)$ , we must have

$$H_{k, N_0}(s) \geq \Psi_1(s) + \frac{1}{2} \int_{1-1/s}^{1-1/s'} \frac{h_{k+1, N_0}(s't)}{t(1-t)} dt + H_{k, N_0}(s') - \varepsilon.$$

Making  $N_0 \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$  yields

$$H_k(s) \geq \Psi_1(s) + \frac{1}{2} \int_{1-1/s}^{1-1/s'} \frac{h_{k+1}(s't)}{t(1-t)} dt + H_k(s').$$

Now it remains to take  $k \rightarrow \infty$  to get the desired result. □

**Lemma 5.2.** For  $5 \geq s' \geq 3 \geq s \geq 2$ ,  $s' - s'/s \geq 2$  and  $s \leq \kappa_3 < \kappa_2 < \kappa_1 \leq s'$ , we have

$$\begin{aligned} H(s) &\geq \Psi_2(s) + \frac{4}{5}H(s') + \frac{1}{5}H(\kappa_1) + \frac{1}{5} \int_{1-1/s}^{1-1/s'} \frac{h(s't)}{t(1-t)} dt \\ &\quad + \frac{1}{5} \int_{1-1/\kappa_2}^{1-1/s'} \frac{h(s't)}{t(1-t)} dt + \frac{1}{5} \int_{1-1/\kappa_3}^{1-1/s'} \frac{h(s't)}{t(1-t)} dt \\ &\quad + \frac{1}{5} \int_{1/s'}^{1/\kappa_2} \frac{dt}{t} \int_t^{1/\kappa_2} \frac{H(s' - s't - s'u)}{u(1-t-u)} du \\ &\quad + \frac{1}{5} \int_{1/s'}^{1/\kappa_1} \frac{dt}{t} \int_{1/\kappa_2}^{1/\kappa_3} \frac{H(s' - s't - s'u)}{u(1-t-u)} du \\ &\quad + \frac{1}{5} \int_{1/s'}^{1/\kappa_1} \frac{dt}{t} \int_t^{1/\kappa_2} \frac{H((1-t-u)/t)}{u(1-t-u)} du, \end{aligned}$$

where  $\Psi_2(s) = \Psi_2(s, s', \kappa_1, \kappa_2, \kappa_3)$  is given by

$$\begin{aligned} \Psi_2(s) := & -\frac{2}{5} \int_2^{s'-1} \frac{\log(t-1)}{t} dt - \frac{2}{5} \int_2^{\kappa_1-1} \frac{\log(t-1)}{t} dt - \frac{1}{5} \int_2^{\kappa_2-1} \frac{\log(t-1)}{t} dt \\ & + \frac{1}{5} \int_{1-1/s}^{1-1/s'} \frac{\log(s't-1)}{t(1-t)} dt + \frac{1}{5} \int_{1-1/\kappa_3}^{1-1/\kappa_1} \frac{\log(\kappa_1 t-1)}{t(1-t)} dt - \frac{2}{5} \sum_{i=9}^{21} I_{2,i}(s) \end{aligned}$$

and  $I_{2,i}(s) = I_{2,i}(s, s', \kappa_1, \kappa_2, \kappa_3)$  is given by

$$\begin{aligned} I_{2,i}(s) &:= \max_{\phi \geq 2} \int_{\mathbb{D}_{2,i}} \omega\left(\frac{\phi-t-u-v}{u}\right) \frac{dt du dv}{tu^2 v} \quad (i = 9, \dots, 15), \\ I_{2,i}(s) &:= \max_{\phi \geq 2} \int_{\mathbb{D}_{2,i}} \omega\left(\frac{\phi-t-u-v-w}{v}\right) \frac{dt du dv dw}{tuv^2 w} \quad (i = 16, \dots, 19), \\ I_{2,20}(s) &:= \max_{\phi \geq 2} \int_{\mathbb{D}_{2,20}} \omega\left(\frac{\phi-t-u-v-w-x}{w}\right) \frac{dt du dv dw dx}{tuvw^2 x}, \\ I_{2,21}(s) &:= \max_{\phi \geq 2} \int_{\mathbb{D}_{2,21}} \omega\left(\frac{\phi-t-u-v-w-x-y}{x}\right) \frac{dt du dv dw dx dy}{tuvwx^2 y}. \end{aligned}$$

The sets  $\mathbb{D}_{2,i}$  ( $9 \leq i \leq 21$ ) are defined as follows:

$$\begin{aligned} \mathbb{D}_{2,9} &:= \{(t, u, v) : 1/\kappa_1 \leq t \leq u \leq v \leq 1/\kappa_3\}, \\ \mathbb{D}_{2,10} &:= \{(t, u, v) : 1/\kappa_1 \leq t \leq u \leq 1/\kappa_2 \leq v \leq 1/s\}, \\ \mathbb{D}_{2,11} &:= \{(t, u, v) : 1/\kappa_1 \leq t \leq 1/\kappa_2 \leq u \leq v \leq 1/\kappa_3\}, \\ \mathbb{D}_{2,12} &:= \{(t, u, v) : 1/s' \leq t \leq u \leq 1/\kappa_1, 1/\kappa_3 \leq v \leq 1/s\}, \\ \mathbb{D}_{2,13} &:= \{(t, u, v) : 1/s' \leq t \leq 1/\kappa_1 \leq u \leq 1/\kappa_2 \leq v \leq 1/s\}, \\ \mathbb{D}_{2,14} &:= \{(t, u, v) : 1/s' \leq t \leq 1/\kappa_1, 1/\kappa_2 \leq u \leq v \leq 1/s\}, \\ \mathbb{D}_{2,15} &:= \{(t, u, v) : 1/\kappa_1 \leq t \leq 1/\kappa_2 \leq u \leq 1/\kappa_3 \leq v \leq 1/s\}, \\ \mathbb{D}_{2,16} &:= \{(t, u, v, w) : 1/\kappa_2 \leq t \leq u \leq v \leq w \leq 1/\kappa_3\}, \\ \mathbb{D}_{2,17} &:= \{(t, u, v, w) : 1/\kappa_2 \leq t \leq u \leq v \leq 1/\kappa_3 \leq w \leq 1/s\}, \\ \mathbb{D}_{2,18} &:= \{(t, u, v, w) : 1/\kappa_2 \leq t \leq u \leq 1/\kappa_3 \leq v \leq w \leq 1/s\}, \\ \mathbb{D}_{2,19} &:= \{(t, u, v, w) : 1/\kappa_1 \leq t \leq 1/\kappa_2, 1/\kappa_3 \leq u \leq v \leq w \leq 1/s\}, \\ \mathbb{D}_{2,20} &:= \{(t, u, v, w, x) : 1/\kappa_2 \leq t \leq 1/\kappa_3 \leq u \leq v \leq w \leq x \leq 1/s\}, \\ \mathbb{D}_{2,21} &:= \{(t, u, v, w, x, y) : 1/\kappa_3 \leq t \leq u \leq v \leq w \leq x \leq y \leq 1/s\}. \end{aligned}$$

*Proof.* By (3.3) of Lemma 3.2, we have

$$(5.11) \quad \sum_d \sigma(d) \Gamma_1 \leq \{4A(s') + A(\kappa_1) - 4H_{k,N_0}(s') - H_{k,N_0}(\kappa_1)\} \Theta(N, \sigma).$$

Similar to (3.20), we can prove

$$(5.12) \quad \sum_d \sigma(d) \Gamma_2 \geq \left\{ \int_{1-1/s}^{1-1/s'} \frac{a(s't) + h_{k+1,N_0}(s't)}{t(1-t)} dt - \varepsilon \right\} \Theta(N, \sigma),$$

$$(5.13) \quad \sum_d \sigma(d) \Gamma_3 \geq \left\{ \int_{1-1/\kappa_2}^{1-1/s'} \frac{a(s't) + h_{k+1,N_0}(s't)}{t(1-t)} dt - \varepsilon \right\} \Theta(N, \sigma),$$

$$(5.14) \quad \sum_d \sigma(d) \Gamma_4 \geq \left\{ \int_{1-1/\kappa_3}^{1-1/s'} \frac{a(s't) + h_{k+1,N_0}(s't)}{t(1-t)} dt - \varepsilon \right\} \Theta(N, \sigma).$$

Similar to (3.20) and in view of  $A(s' - s't - s'u) = A((1 - t - u)/t) = 1$ , we have

$$(5.15) \quad \sum_d \sigma(d) \Gamma_5 \leq \left\{ \int_{1/s'}^{1/\kappa_2} \frac{dt}{t} \int_t^{1/\kappa_2} \frac{1 - H_{k+2, N_0}(s' - s't - s'u)}{u(1 - t - u)} du + \varepsilon \right\} \Theta(N, \sigma),$$

$$(5.16) \quad \sum_d \sigma(d) \Gamma_6 \leq \left\{ \int_{1/s'}^{1/\kappa_1} \frac{dt}{t} \int_{1/\kappa_2}^{1/\kappa_3} \frac{1 - H_{k+2, N_0}(s' - s't - s'u)}{u(1 - t - u)} du + \varepsilon \right\} \Theta(N, \sigma),$$

$$(5.17) \quad \sum_d \sigma(d) \Gamma_7 \leq \left\{ \int_{1/s'}^{1/\kappa_1} \frac{dt}{t} \int_t^{1/\kappa_1} \frac{1 - H_{k+2, N_0}((1 - t - u)/t)}{u(1 - t - u)} du + \varepsilon \right\} \Theta(N, \sigma),$$

$$(5.18) \quad \sum_d \sigma(d) \Gamma_8 \leq \left\{ \int_{1/s'}^{1/\kappa_1} \frac{dt}{t} \int_{1/\kappa_1}^{1/\kappa_2} \frac{1 - H_{k+2, N_0}((1 - t - u)/t)}{u(1 - t - u)} du + \varepsilon \right\} \Theta(N, \sigma).$$

We have also, for  $i = 9, \dots, 21$ ,

$$(5.19) \quad \sum_d \sigma(d) \Gamma_i \leq \{2I_{2,i}(s) + \varepsilon\} \Theta(N, \sigma).$$

As before, inserting (5.11)–(5.19) into the inequality of Lemma 4.2 and using the definition of  $H_{k, N_0}(s)$ , we can deduce

$$(5.20) \quad 5H_{k, N_0}(s) \geq A(s, s') + B(s, s') - \varepsilon,$$

where

$$\begin{aligned} A(s, s') &:= 5A(s) - 4A(s') - A(\kappa_1) + 4H_{k, N_0}(s') + H_{k, N_0}(\kappa_1) \\ &+ \int_{1-1/s}^{1-1/s'} \frac{a(s't)}{t(1-t)} dt + \int_{1-1/\kappa_2}^{1-1/s'} \frac{a(s't)}{t(1-t)} dt + \int_{1-1/\kappa_3}^{1-1/s'} \frac{a(s't)}{t(1-t)} dt \\ &- \int_{1/s'}^{1/\kappa_2} \int_t^{1/\kappa_2} \frac{dt du}{tu(1-t-u)} - \int_{1/s'}^{1/\kappa_1} \int_t^{1/\kappa_3} \frac{dt du}{tu(1-t-u)} - 2 \sum_{i=9}^{21} I_{2,i}(s) \end{aligned}$$

and

$$\begin{aligned} B(s, s') &:= \left( \int_{1-1/s}^{1-1/s'} + \int_{1-1/\kappa_2}^{1-1/s'} + \int_{1-1/\kappa_3}^{1-1/s'} \right) \frac{h_{k+1, N_0}(s't)}{t(1-t)} dt \\ &+ \int_{1/s'}^{1/\kappa_2} \frac{dt}{t} \int_t^{1/\kappa_2} \frac{H_{k+2, N_0}(s' - s't - s'u)}{u(1-t-u)} du \\ &+ \int_{1/s'}^{1/\kappa_1} \frac{dt}{t} \int_{1/\kappa_2}^{1/\kappa_3} \frac{H_{k+2, N_0}(s' - s't - s'u)}{u(1-t-u)} du \\ &+ \int_{1/s'}^{1/\kappa_1} \frac{dt}{t} \int_t^{1/\kappa_2} \frac{H_{k+2, N_0}((1-t-u)/t)}{u(1-t-u)} du. \end{aligned}$$

For  $a \geq b > 2$ , we have

$$\begin{aligned} \int_{1/a}^{1/b} \frac{dt}{t} \int_t^{1/b} \frac{du}{u(1-t-u)} &= \int_{1/a}^{1/b} \frac{du}{u} \int_{1/a}^u \frac{dt}{t(1-t-u)} \\ &= \int_{1/a}^{1/b} \frac{\log(a-1-au) - \log(1/u-2)}{u(1-u)} du \\ &= \int_{1-1/b}^{1-1/a} \frac{\log(at-1)}{t(1-t)} dt - \int_{b-1}^{a-1} \frac{\log(t-1)}{t} dt, \end{aligned}$$

where we have used the change of variables  $t = 1 - u$  and  $t = 1/u - 1$  respectively.

Similarly for  $a \geq b \geq c \geq d > 2$ , we have

$$\int_{1/a}^{1/b} \frac{dt}{t} \int_{1/c}^{1/d} \frac{du}{u(1-t-u)} = \int_{1-1/d}^{1-1/c} \frac{\log(at-1)}{t(1-t)} dt - \int_{1-1/d}^{1-1/c} \frac{\log(bt-1)}{t(1-t)} dt.$$

By using these two relations and (5.10), a simple calculation shows

$$A(s, s') = 5\Psi_2(s) + 4H_{k, N_0}(s') + H_{k, N_0}(\kappa_1).$$

Inserting this into (5.20) and making  $N \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$  and  $k \rightarrow \infty$ , we obtain the desired inequality. This completes the proof.  $\square$

## § 6. Proofs of Propositions 3 and 4

We first prove a preliminary lemma. Let  $\mathbf{1}_{[a,b]}(t)$  be the characteristic function of the interval  $[a, b]$ . We put

$$\sigma(a, b, c) := \int_a^b \log \frac{c}{t-1} \frac{dt}{t}, \quad \sigma_0(t) := \frac{\sigma(3, t+2, t+1)}{1 - \sigma(3, 5, 4)}.$$

**Lemma 6.1.** *Let  $3 \leq s' \leq 5$ ,  $0 < a < b < 1$  and  $2 \leq ac < bc \leq 4$ . Then we have*

$$(6.1) \quad h(4) \geq \int_1^3 H(t) \frac{\sigma_0(t)}{t} dt.$$

$$(6.2) \quad H(s') \geq \int_1^3 H(t) \left\{ \frac{\sigma_0(t)}{t} \log \left( \frac{4}{s'-1} \right) + \frac{\mathbf{1}_{[s'-2, 3]}(t)}{t} \log \left( \frac{t+1}{s'-1} \right) \right\} dt,$$

$$(6.3) \quad \int_a^b \frac{h(ct)}{t(1-t)} dt \geq \log \left( \frac{b-ab}{a-ab} \right) \int_1^3 H(t) \frac{\sigma_0(t) + \mathbf{1}_{[bc-1, 3]}(t)}{t} dt \\ + \int_1^3 H(t) \frac{\mathbf{1}_{[ac-1, bc-1]}(t)}{t} \log \left( \frac{(1-a)(t+1)}{a(c-1-t)} \right) dt.$$

*Proof.* By Proposition 2, we have

$$(6.4) \quad H(s') \geq \int_{s'-1}^4 \frac{h(u)}{u} du \geq \int_{s'-1}^4 \left( h(4) + \int_{u-1}^3 H(t) \frac{dt}{t} \right) \frac{du}{u} \\ = h(4) \log \left( \frac{4}{s'-1} \right) + \int_1^3 H(t) \frac{\mathbf{1}_{[s'-2, 3]}(t)}{t} \log \left( \frac{t+1}{s'-1} \right) dt.$$

From Proposition 2 and (6.4), we deduce

$$h(4) \geq \int_3^5 \frac{H(v)}{v} dv \geq h(4) \sigma(3, 5, 4) + \int_1^3 H(t) \frac{\sigma(3, t+2, t+1)}{t} dt,$$

which implies the inequality (6.1).

The inequality (6.2) follows immediately from (6.4) and (6.1).

By using Proposition 2, we have

$$\int_a^b \frac{h(ct)}{t(1-t)} dt = c \int_{ac}^{bc} \frac{h(u)}{u(c-u)} du \geq c \int_{ac}^{bc} \frac{du}{u(c-u)} \left( h(4) + \int_{u-1}^3 H(t) \frac{dt}{t} \right) \\ = \left\{ h(4) + \int_1^3 H(t) \frac{\mathbf{1}_{[bc-1, 3]}(t)}{t} dt \right\} \log \left( \frac{b-ab}{a-ab} \right) \\ + \int_1^3 H(t) \frac{\mathbf{1}_{[ac-1, bc-1]}(t)}{t} \log \left( \frac{(1-a)(t+1)}{a(c-1-t)} \right) dt,$$

which combines (6.1) to give (6.3). This completes the proof.  $\square$

*Proof of Proposition 3.* By using Lemma 6.1, a simple calculation shows

$$\frac{1}{2} \int_{1-1/s}^{1-1/s'} \frac{h(s't)}{t(1-t)} dt + H(s') \geq \int_1^3 H(t) \Xi_1(t; s) dt,$$

which, together with Lemma 4.1, implies the desired result.  $\square$

*Proof of Proposition 4.* From (6.1)–(6.3), we can deduce

$$\begin{aligned} (6.5) \quad & \int_{1-1/s}^{1-1/s'} \frac{h(s't)}{t(1-t)} dt + \int_{1-1/\kappa_2}^{1-1/s'} \frac{h(s't)}{t(1-t)} dt + \int_{1-1/\kappa_3}^{1-1/s'} \frac{h(s't)}{t(1-t)} dt + 4H(s') + H(\kappa_1) \\ & \geq \int_1^3 H(t) \frac{\mathbf{1}_{[\alpha_2, 3]}(t)}{t} \log \left( \frac{(t+1)^5}{(s-1)(s'-1)(\kappa_1-1)(\kappa_2-1)(\kappa_3-1)} \right) dt \\ & \quad + \int_1^3 H(t) \frac{\sigma_0(t)}{t} \log \left( \frac{1024}{(s-1)(s'-1)(\kappa_1-1)(\kappa_2-1)(\kappa_3-1)} \right) dt \\ & \quad + \int_1^3 H(t) \frac{\mathbf{1}_{[\alpha_5, \alpha_2]}(t)}{t} \log \left( \frac{t+1}{(\kappa_3-1)(s'-1-t)} \right) dt \\ & \quad + \int_1^3 H(t) \frac{\mathbf{1}_{[\alpha_4, \alpha_2]}(t)}{t} \log \left( \frac{t+1}{(\kappa_2-1)(s'-1-t)} \right) dt \\ & \quad + \int_1^3 H(t) \frac{\mathbf{1}_{[\alpha_3, \alpha_2]}(t)}{t} \log \left( \frac{t+1}{(s-1)(s'-1-t)} \right) dt \\ & \quad + \int_1^3 H(t) \frac{\mathbf{1}_{[\alpha_1, \alpha_2]}(t)}{t} \log \left( \frac{t+1}{\kappa_1-1} \right) dt. \end{aligned}$$

By the change of variable  $v = s'(1-t-u)$ , we have

$$\int_{1/s'}^{1/\kappa_2} \frac{dt}{t} \int_t^{1/\kappa_2} \frac{H(s' - s't - s'u)}{u(1-t-u)} du = \int_{1/s'}^{1/\kappa_2} \frac{dt}{t} \int_{s'(1-1/\kappa_2-t)}^{s'(1-2t)} \frac{s'H(v)}{v(s' - s't - v)} dv.$$

Interchanging the order of integration and a simple calculation show that

$$\begin{aligned} (6.6) \quad & \int_{1/s'}^{1/\kappa_2} \frac{dt}{t} \int_t^{1/\kappa_2} \frac{H(s' - s't - s'u)}{u(1-t-u)} du = \int_1^3 H(t) \left\{ \frac{\mathbf{1}_{[\alpha_6, \alpha_4]}(t)}{t(1-t/s')} \log \left( \frac{s'}{\kappa_2 s' - s' - \kappa_2 t} \right) \right. \\ & \quad \left. + \frac{\mathbf{1}_{[\alpha_4, \alpha_2]}(t)}{t(1-t/s')} \log(s' - 1 - t) \right\} dt. \end{aligned}$$

Similarly we can prove

$$(6.7) \quad \int_{1/s'}^{1/\kappa_1} \frac{dt}{t} \int_{1/\kappa_2}^{1/\kappa_3} \frac{H(s' - s't - s'u)}{u(1-t-u)} du = \int_1^3 H(t) L_1(t) dt$$

where

$$\begin{aligned} L_1(t) := & \frac{\mathbf{1}_{[\alpha_7, \alpha_5]}(t)}{t(1-t/s')} \log \left( \frac{s'^2}{(\kappa_1 s' - s' - \kappa_1 t)(\kappa_3 s' - s' - \kappa_3 t)} \right) \\ & + \frac{\mathbf{1}_{[\alpha_5, \alpha_8]}(t)}{t(1-t/s')} \log \left( \frac{s'(s' - 1 - t)}{\kappa_1 s' - s' - \kappa_1 t} \right) \\ & + \frac{\mathbf{1}_{[\alpha_8, \alpha_4]}(t)}{t(1-t/s')} \log \left( \frac{(s' - 1 - t)(\kappa_2 s' - s' - \kappa_2 t)}{s'} \right), \end{aligned}$$

and

$$(6.8) \quad \int_{1/s'}^{1/\kappa_1} \frac{dt}{t} \int_t^{1/\kappa_2} \frac{H((1-t-u)/t)}{u(1-t-u)} du = \int_1^3 H(t) L_2(t) dt$$

with

$$L_2(t) := \frac{\mathbf{1}_{[\alpha_9, \alpha_1]}(t)}{t} \log \left( \frac{t+1}{(\kappa_2-1)(\kappa_1-1-t)} \right) \\ + \frac{\mathbf{1}_{[\alpha_1, \alpha_4]}(t)}{t} \log \left( \frac{t+1}{\kappa_2-1} \right) + \frac{\mathbf{1}_{[\alpha_4, \alpha_2]}(t)}{t} \log(s'-1-t).$$

Now by inserting (6.5)–(6.8) into Lemma 5.2, we easily deduce the required result.  $\square$

## § 7. Proof of Theorem 1

We need to resolve the functional inequalities (3.21) and (3.22). It seems very difficult to give the exact solutions, because we only know that  $H(s)$  is decreasing. Next we shall give a numeric lower bound for solution by using discretion, which is sufficient to prove Theorem 1.

Put  $s_0 := 1$  and  $s_i := 2 + 0.1 \times (i + 1)$  for  $i = 1, \dots, 9$ . Since  $H(s)$  is decreasing on  $[1, 10]$ , Proposition 4 allows us to deduce

$$(7.1) \quad H(s_i) \geq \Psi_2(s_i) + \sum_{j=1}^9 a_{i,j} H(s_j),$$

where

$$a_{i,j} := \int_{s_{j-1}}^{s_j} \Xi_2(t, s_i) dt \quad (i = 1, \dots, 4; j = 1, \dots, 9).$$

Similarly Proposition 3 implies

$$(7.2) \quad H(s_i) \geq \Psi_1(s_i) + \sum_{j=1}^9 a_{i,j} H(s_j),$$

where

$$a_{i,j} := \int_{s_{j-1}}^{s_j} \Xi_1(t, s_i) dt \quad (i = 5, \dots, 9; j = 1, \dots, 9)$$

**Table 1.** Choice of parameters

$i$	$s_i$	$s'_i$	$\kappa_{1,i}$	$\kappa_{2,i}$	$\kappa_{3,i}$	$\Psi_1(s_i)$	$\Psi_2(s_i)$
1	2.2	4.54	3.53	2.90	2.44		0.015826357
2	2.3	4.50	3.54	2.88	2.43		0.015247971
3	2.4	4.46	3.57	2.87	2.40		0.013898757
4	2.5	4.12	3.56	2.91	2.50		0.011776059
5	2.6	3.58				0.009405211	
6	2.7	3.47				0.006558950	
7	2.8	3.34				0.003536751	
8	2.9	3.19				0.001056651	
9	3.0	3.00				0.000000000	

The parameters  $s'_i$ ,  $\kappa_{1,i}$ ,  $\kappa_{2,i}$  and  $\kappa_{3,i}$  are chosen such that  $\Psi_1(s_i)$  or  $\Psi_2(s_i)$  is maximal.

We put

$$\mathbf{A} := \begin{pmatrix} a_{1,1} & \cdots & a_{1,9} \\ \vdots & & \vdots \\ a_{9,1} & \cdots & a_{9,9} \end{pmatrix}, \quad \mathbf{H} := \begin{pmatrix} H(s_1) \\ \vdots \\ H(s_9) \end{pmatrix}, \quad \mathbf{B} := \begin{pmatrix} \Psi_2(s_1) \\ \vdots \\ \Psi_2(s_4) \\ \Psi_1(s_5) \\ \vdots \\ \Psi_1(s_9) \end{pmatrix}.$$

Then (7.1) and (7.2) can be written as

$$(7.3) \quad (\mathbf{I} - \mathbf{A})\mathbf{H} \geq \mathbf{B},$$

where  $\mathbf{I}$  is the unit matrix.

In order to resolve (7.3), we first solve the system of linear equations

$$(7.4) \quad (\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{B},$$

by using *Maple* and obtain

$$\mathbf{X} = \begin{pmatrix} 0.0223939 \cdots \\ 0.0217196 \cdots \\ 0.0202876 \cdots \\ 0.0181433 \cdots \\ 0.0158644 \cdots \\ 0.0129923 \cdots \\ 0.0100686 \cdots \\ 0.0078162 \cdots \\ 0.0072943 \cdots \end{pmatrix}.$$

From (7.3) and (7.4), we deduce that

$$(\mathbf{I} - \mathbf{A})(\mathbf{H} - \mathbf{X}) \geq \mathbf{0}.$$

Since all elements of  $(\mathbf{I} - \mathbf{A})^{-1}$  are positive, it follows that

$$\mathbf{H} \geq \mathbf{X}.$$

In particular we have

$$H(2.2) \geq 0.0223939.$$

Now taking  $\sigma = \{1\}$  and  $s = 2.2$  in (3.3) of Lemma 3.2, we find, for  $\delta$  sufficiently small,  $N_0$  sufficiently large and  $N \geq N_0$ ,

$$\begin{aligned} D(N) &\leq S(\mathcal{A}; \mathcal{P}(N), N^{(1/2-\delta)/2.2}) = \Phi(N, \{1\}, 2.2) \\ &\leq \{A(2.2) - H_{k, N_0}(2.2)\} \frac{4C_N \text{li}(N)}{\log(N^{1/2-\delta})} \\ &\leq 8(1 - 0.0223938)\Theta(N) \\ &\leq 7.82085\Theta(N). \end{aligned}$$

This completes the proof of Theorem 1.



**Remark 2.** (i) The constant  $s_1 = 2.2$  comes from the fact that  $\Psi_2(s)$  attains the maximal value at  $s = s_1$  (approximately). Since  $H(s)$  is decreasing on  $[1, 10]$ , we have  $H(2.1) \geq 0.0223939$ . In order to obtain a better lower (which leads to a smaller constant than 7.82085), we must look for a new weighted inequality (as in Lemma 4.1 and 4.2) such that the corresponding main term  $\Psi(2.1)$  has a larger lower bound than 0.015826357.

(ii) If we divide the interval  $[2, 3]$  into more subintervals than 9, it is certain that we can obtain a better result. But the improvement is very minuscule.

### § 8. Proof of Theorem 3

In the case of the twin primes problem, we need to sieve the following sequence

$$\mathcal{B} := \{p + 2 : p \leq x\}.$$

Thinking to Lemmas 2.7 and 2.9, we have  $\frac{4}{7}$  for the level of distribution in place  $\frac{1}{2}$  in the Bombieri–Vinogradov theorem. Thus we can take  $Q := x^{4/7-\delta}$  and  $\underline{d} := Q/d$  in the definitions described in Section 3. As before, we can prove the corresponding Propositions 3 and 4 with the following modification: In the definition of  $\Psi_1(s)$  we add a factor  $\frac{7}{8}$  before  $I_1(s)$ , and in the definition of  $\Psi_2(s)$  we replace the factor  $\frac{2}{5}$  before the sum by  $\frac{7}{20}$ . When we use the switching principle to treat the terms  $\Omega_3$  and  $\Gamma_i$  for  $5 \leq i \leq 21$ , the related error terms can be estimated by using Lemma 2.9 which has  $\frac{4}{7}$  for the level of distribution (see [26], page 380).

**Table 2.** Choice of parameters

$i$	$s_i$	$s'_i$	$\kappa_{1,i}$	$\kappa_{2,i}$	$\kappa_{3,i}$	$\Psi_1(s_i)$	$\Psi_2(s_i)$
1	2.1	4.93	3.62	2.86	2.34		0.020914508
2	2.2	4.91	3.62	2.85	2.33		0.020399717
3	2.3	5.00	3.63	2.82	2.30		0.019005124
4	2.4	4.52	3.64	2.87	2.40		0.016618139
5	2.5	3.72				0.013597508	
6	2.6	3.62				0.010644985	
7	2.7	3.49				0.007155027	
8	2.8	3.35				0.003741586	
9	2.9	3.19				0.001087780	
10	3.0	3.00				0.000000000	

As before we can prove

$$H(2.1) \geq 0.0287118.$$

Thus for  $\delta$  sufficiently small,  $x_0$  sufficiently large and  $x \geq x_0$ , we have

$$\begin{aligned} \pi_2(x) &\leq S(\mathcal{B}; \mathcal{P}(2), x^{(1/2-\delta)/2.1}) \\ &\leq 3.5(1 - 0.0287117) \Pi(x) \\ &\leq 3.39951 \Pi(x). \end{aligned}$$

This completes the proof of Theorem 3.

## § 9. Chen's system of weights

Let

$$\mathcal{A} := \{N - p : p \leq N\}, \quad \mathcal{P}(q) := \{p : (p, q) = 1\}.$$

The inequality (9.1) below appeared in [9] (page 479, (11)) with  $(\kappa, \sigma) = (\frac{1}{12}, \frac{1}{3.047})$ ,  $(\frac{1}{9.2}, \frac{1}{3.41})$  without proof. Cai [3] gave a proof with an extra assumption  $3\sigma + \kappa > 1$ . Here we present a proof without Cai's assumption. This removal is important in our argument.

**Lemma 9.1.** *Let  $0 < \kappa < \sigma < \frac{1}{3}$ . Then we have*

$$(9.1) \quad D_{1,2}(N) \geq S(\mathcal{A}; \mathcal{P}(N), N^\kappa) - \frac{1}{2}S_1 - S_2 - \frac{1}{2}S_3 + \frac{1}{2}S_4 + O(N^{1-\kappa}),$$

where  $S_i = S_i(\kappa, \sigma)$  ( $1 \leq i \leq 4$ ) are defined by

$$\begin{aligned} S_1 &:= \sum_{\substack{N^\kappa \leq p < N^\sigma \\ (p, N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^\kappa), \\ S_2 &:= \sum_{\substack{N^\sigma \leq p_1 < p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} \sum_{p_2} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N p_1), p_2), \\ S_3 &:= \sum_{\substack{N^\kappa \leq p_1 < N^\sigma \leq p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} \sum_{p_2} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N p_1), p_2), \\ S_4 &:= \sum_{\substack{N^\kappa \leq p_1 < p_2 < p_3 < N^\sigma \\ (p_1 p_2 p_3, N)=1}} \sum_{p_2} \sum_{p_3} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N p_1), p_2). \end{aligned}$$

*Proof.* Clearly the desired inequality (9.1) is equivalent to

$$(9.2) \quad D_{1,2}(N) \geq \sum_{a \in \mathcal{A}, (a, P(N^\kappa))=1} \left(1 - \frac{1}{2}s_1(a) - s_2(a) - \frac{1}{2}s_3(a) + \frac{1}{2}s_4(a)\right) + O(N^{1-\kappa}),$$

where

$$\begin{aligned} s_1(a) &:= \sum_{\substack{N^\kappa \leq p < N^\sigma \\ p|a, (p, N)=1}} 1, \\ s_2(a) &:= \sum_{\substack{N^\sigma \leq p_1 < p_2 < (N/p_1)^{1/2} \\ p_1 p_2 | a, (p_1 p_2, N)=1 \\ p|(a/p_1 p_2) \Rightarrow p \geq p_2}} \sum_{p_2} 1, \\ s_3(a) &:= \sum_{\substack{N^\kappa \leq p_1 < N^\sigma \leq p_2 < (N/p_1)^{1/2} \\ p_1 p_2 | a, (p_1 p_2, N)=1 \\ p|(a/p_1 p_2) \Rightarrow p \geq p_2}} \sum_{p_2} 1, \\ s_4(a) &:= \sum_{\substack{N^\kappa \leq p_1 < p_2 < p_3 < N^\sigma \\ p_1 p_2 p_3 | a, (p_1 p_2 p_3, N)=1 \\ p|(a/p_1 p_2 p_3) \Rightarrow p \geq p_2}} \sum_{p_2} \sum_{p_3} 1. \end{aligned}$$

Let

$$\delta^*(a) := \begin{cases} 1 & \text{if } \Omega(a) \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see

$$D_{1,2}(N) \geq \sum_{a \in \mathcal{A}, (a, P(N^\kappa))=1} \delta^*(a) = \sum_{a \in \mathcal{A}, (a, P(N^\kappa))=1} \mu(a)^2 \delta^*(a) + O(N^{1/2}),$$

where we have used the fact that

$$\sum_{\substack{a \in \mathcal{A} \\ (a, P(N^\kappa))=1}} \{1 - \mu(a)^2\} \delta^*(a) \leq \sum_{N^\kappa \leq p \leq N^{1/2}} 1 \ll N^{1/2}.$$

Similarly if we write

$$\delta(a) := 1 - \frac{1}{2}s_1(a) - s_2(a) - \frac{1}{2}s_3(a) + \frac{1}{2}s_4(a),$$

we can show that

$$\sum_{a \in \mathcal{A}, (a, P(N^\kappa))=1} \delta(a) = \sum_{a \in \mathcal{A}, (a, P(N^\kappa))=1} \mu(a)^2 \delta(a) + O(N^{1-\kappa}).$$

Thus in order to prove (9.2) it suffices to verify that

$$(9.3) \quad \delta^*(a) \geq \delta(a)$$

for  $a \in \mathcal{A}$ ,  $\mu(a)^2 = 1$  and  $(a, P(N^\kappa)) = (a, N) = 1$ .

We first observe that (9.3) is trivial if  $\Omega(a) \leq 2$ , since  $\delta^*(a) = 1$  and  $s_4(a) = 0$  in this case. It remains to show that  $\delta(a) \leq 0$  in all other cases, which can be verified as follows:

If  $\Omega(a) = 3$  and  $s_1(a) = 0$ , then  $s_2(a) = 1$  and  $s_3(a) = s_4(a) = 0$ . Thus  $\delta(a) = 0$ .

If  $\Omega(a) = 3$  and  $s_1(a) = 1$ , then  $s_3(a) = 1$  and  $s_2(a) = s_4(a) = 0$ . Thus  $\delta(a) = 0$ .

If  $\Omega(a) = 3$  and  $s_1(a) = 2$ , then  $s_2(a) = s_3(a) = s_4(a) = 0$ . Thus  $\delta(a) = 0$ .

If  $\Omega(a) = 3$  and  $s_1(a) = 3$ , then  $s_2(a) = s_3(a) = 0$  and  $s_4(a) = 1$ . Thus  $\delta(a) = 0$ .

If  $\Omega(a) \geq 4$  and  $s_1(a) = 1$ , then  $s_3(a) = 1$  and  $s_2(a) = s_4(a) = 0$ . Thus  $\delta(a) = 0$ .

If  $\Omega(a) \geq 4$  and  $s_1(a) = 2$ , then  $s_2(a) = s_3(a) = s_4(a) = 0$ . Thus  $\delta(a) = 0$ .

If  $\Omega(a) \geq 4$  and  $s_1(a) \geq 3$ , then  $s_2(a) = s_3(a) = 0$  and  $s_4(a) = s_1(a) - 2$ . Thus  $\delta(a) = 0$ .

This completes the proof.  $\square$

The main difference between (9.1) and Chen's other weighted inequalities (see (34) of [7] and page 425 of [8]) is the additional positive term  $S_4$ . However a direct application of sieve to  $S_4$  leads to zero contribution. In order to take advantage of  $S_4$ , Chen used (9.1) with two different couples of parameters  $(\kappa, \sigma)$ . Then an agreeable application of the Buchstab identity and switching principle leads to some compensation. This idea was also used by Cai & Lu [4] and Cai [3]. Here we make some modifications of their argument such that this process is more powerful.

**Lemma 9.2.** *Let  $0 < \kappa_1 < \kappa_2 < \rho < \sigma_2 < \sigma_1 < \frac{1}{3}$  such that  $3\kappa_1 + \rho \geq \frac{1}{2}$ . Then we have*

$$(9.4) \quad 4D_{1,2}(N) \geq 4S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_1}) - \Upsilon_1 - \Upsilon_2 - \Upsilon_3 + \Upsilon_4 + \Upsilon_5 + \Upsilon_6 - 2\Upsilon_7 \\ - 2\Upsilon_8 - \Upsilon_9 - \Upsilon_{10} + \Upsilon_{11} + \Upsilon_{12} - \Upsilon_{13} - \Upsilon_{14} + \Upsilon_{15} + O(N^{1-\kappa_1}),$$

where

$$\begin{aligned}
\Upsilon_1 &:= \sum_{\substack{N^{\kappa_1} \leq p < N^{\kappa_2} \\ (p, N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), p), \\
\Upsilon_2 &:= \sum_{\substack{N^{\kappa_1} \leq p < N^{\sigma_1} \\ (p, N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa_1}), \\
\Upsilon_3 &:= \sum_{\substack{N^{\kappa_1} \leq p < N^{\sigma_2} \\ (p, N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa_1}), \\
\Upsilon_4 &:= \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < N^{\kappa_2} \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), N^{\kappa_1}), \\
\Upsilon_5 &:= \sum_{\substack{N^{\kappa_1} \leq p_1 < N^{\kappa_2} \leq p_2 < N^{1/2-2\kappa_1}/p_1 \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), N^{\kappa_1}), \\
\Upsilon_6 &:= \sum_{\substack{N^{\kappa_1} \leq p_1 < N^{\kappa_2}, N^{\rho} \leq p_2 < N^{\sigma_2} \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), p_1), \\
\Upsilon_7 &:= \sum_{\substack{N^{\sigma_1} \leq p_1 < p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N p_1), p_2), \\
\Upsilon_8 &:= \sum_{\substack{N^{\sigma_2} \leq p_1 < p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N p_1), p_2), \\
\Upsilon_9 &:= \sum_{\substack{N^{\kappa_1} \leq p_1 < N^{\sigma_1} \leq p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N p_1), p_2), \\
\Upsilon_{10} &:= \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2} \leq p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} \sum S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N p_1), N^{\sigma_1}), \\
\Upsilon_{11} &:= \sum_{\substack{N^{\kappa_2} \leq p_1 < p_2 < p_3 < N^{\sigma_2} \\ (p_1 p_2 p_3, N)=1}} \sum \sum S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_2), \\
\Upsilon_{12} &:= \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2}, N^{\sigma_1} \leq p_2 < p_3 < (N/p_1)^{1/2} \\ (p_1 p_2 p_3, N)=1}} \sum \sum S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N p_1), p_2), \\
\Upsilon_{13} &:= \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < p_3 < p_4 < N^{\kappa_2} \\ (p_1 p_2 p_3 p_4, N)=1}} \sum \sum \sum S(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(N), p_2), \\
\Upsilon_{14} &:= \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < p_3 < N^{\kappa_2} \leq p_4 < N^{1/2-2\kappa_1}/p_3 \\ (p_1 p_2 p_3 p_4, N)=1}} \sum \sum \sum S(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(N), p_2), \\
\Upsilon_{15} &:= \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2} \leq p_2 < p_3 < p_4 < N^{\sigma_1} \\ (p_1 p_2 p_3 p_4, N)=1}} \sum \sum \sum S(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(N), p_3).
\end{aligned}$$

*Proof.* The inequality (9.1) with  $(\kappa, \sigma) = (\kappa_2, \sigma_2)$  implies

$$(9.5) \quad 2D_{1,2}(N) \geq 2S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_2}) - S_1(\kappa_2, \sigma_2) - 2\Upsilon_8 - S_3(\kappa_2, \sigma_2) + \Upsilon_{11} + O(N^{1-\kappa_2}).$$

Buchstab's identity, when applied three times, gives the equality

$$\begin{aligned} 2S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_2}) &= 2S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_1}) - \Upsilon_1 - \sum_{\substack{N^{\kappa_1} \leq p < N^{\kappa_2} \\ (p, N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa_1}) \\ &\quad + \Upsilon_4 - \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < p_3 < N^{\kappa_2} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_1). \end{aligned}$$

Similarly a simple application of Buchstab's identity yields

$$S_1(\kappa_2, \sigma_2) = \sum_{\substack{N^{\kappa_2} \leq p < N^{\sigma_2} \\ (p, N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa_1}) - \sum_{\substack{N^{\kappa_1} \leq p_1 < N^{\kappa_2} \leq p_2 < N^{\rho} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), p_1) - \Upsilon_6.$$

Clearly  $p_1 < N^{\sigma_2}$  and  $\sigma_2 < \sigma_1 < \frac{1}{3}$  imply that  $N^{\sigma_1} < (N/p_1)^{1/2}$ . Thus by Buchstab's identity, we can write

$$\begin{aligned} S_3(\kappa_2, \sigma_2) &= \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2} \leq p_2 < N^{\sigma_1} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N p_1), N^{\sigma_1}) \\ &\quad + \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2} \leq p_2 < p_3 < N^{\sigma_1} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N p_1), p_3) + O(N^{1-\sigma_2}) \\ &\quad + \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2}, N^{\sigma_1} \leq p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N p_1), N^{\sigma_1}) \\ &\quad - \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2}, N^{\sigma_1} \leq p_2 < p_3 < (N/p_1)^{1/2} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N p_1), p_2) \\ &= \Upsilon_{10} - \Upsilon_{12} + \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2} \leq p_2 < p_3 < N^{\sigma_1} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N p_1), p_3) + O(N^{1-\sigma_2}). \end{aligned}$$

Inserting these into (9.5), we find that

$$(9.6) \quad 2D_{1,2}(N) \geq 2S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_1}) - \Upsilon_1 - \Upsilon_3 + \Upsilon_4 + \Upsilon_6 - 2\Upsilon_8 - \Upsilon_{10} + \Upsilon_{11} + \Upsilon_{12} + \Delta + O(N^{1-\kappa_2}),$$

where

$$\begin{aligned} \Delta &:= - \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < p_3 < N^{\kappa_2} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N p_1), p_1) \\ &\quad - \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2} \leq p_2 < p_3 < N^{\sigma_1} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N p_1), p_3) \\ &\quad + \sum_{\substack{N^{\kappa_1} \leq p_1 < N^{\kappa_2} \leq p_2 < N^{\rho} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), p_1). \end{aligned}$$

Next we shall further decompose  $\Delta$ . In view of  $3\kappa_1 + \rho \geq \frac{1}{2}$ , we have  $N^{\rho} \geq N^{1/2-2\kappa_1}/p_1$  provided  $p_1 \geq N^{\kappa_1}$ . Thus Buchstab's identity allows us to write

$$\begin{aligned} \sum_{\substack{N^{\kappa_1} \leq p_1 < N^{\kappa_2} \leq p_2 < N^{\rho} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), p_1) &\geq \Upsilon_5 \\ &\quad - \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < N^{\kappa_2} \leq p_3 < N^{1/2-2\kappa_1}/p_2 \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_1). \end{aligned}$$

Thus we have

$$\Delta \geq \Upsilon_5 + \Delta_1,$$

where

$$\begin{aligned} \Delta_1 := & - \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < p_3 < N^{\kappa_2} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_1) \\ & - \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < N^{\kappa_2} \leq p_3 < N^{1/2-2\kappa_1}/p_2 \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_1) \\ & - \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2} \leq p_2 < p_3 < N^{\sigma_1} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N p_1), p_3). \end{aligned}$$

Now the inequality (9.6) becomes

$$(9.7) \quad \begin{aligned} 2D_{1,2}(N) \geq & 2S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_1}) - \Upsilon_1 - \Upsilon_3 + \Upsilon_4 + \Upsilon_5 + \Upsilon_6 \\ & - 2\Upsilon_8 - \Upsilon_{10} + \Upsilon_{11} + \Upsilon_{12} + \Delta_1 + O(N^{1-\kappa_2}). \end{aligned}$$

The inequality (9.1) with  $(\kappa, \sigma) = (\kappa_1, \sigma_1)$  gives us

$$(9.8) \quad 2D_{1,2}(N) \geq 2S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_1}) - \Upsilon_2 - 2\Upsilon_7 - \Upsilon_9 + S_4(\kappa_1, \sigma_1) + O(N^{1-\kappa_1}).$$

Adding (9.7) to (9.8) yields

$$(9.9) \quad \begin{aligned} 4D_{1,2}(N) \geq & 4S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_1}) - \Upsilon_1 - \Upsilon_2 - \Upsilon_3 + \Upsilon_4 + \Upsilon_5 + \Upsilon_6 \\ & - 2\Upsilon_7 - 2\Upsilon_8 - \Upsilon_9 - \Upsilon_{10} + \Upsilon_{11} + \Upsilon_{12} + \Delta_2 + O(N^{1-\kappa_1}), \end{aligned}$$

where

$$\Delta_2 := \Delta_1 + \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < p_3 < N^{\sigma_1} \\ (p_1 p_2 p_3, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_2).$$

Clearly all domains of summation in the three terms on the right-hand side of  $\Delta_1$  are distinct and are contained in the domain of summation of the last triple sums on the right-hand side of  $\Delta_2$  (since  $3\kappa_1 + \sigma_1 > 3\kappa_1 + \rho \geq \frac{1}{2}$ ). Therefore we have

$$\begin{aligned} \Delta_2 \geq & - \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < p_3 < N^{\kappa_2} \\ (p_1 p_2 p_3, N)=1}} \{S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_1) - S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_2)\} \\ & - \sum_{\substack{N^{\kappa_1} \leq p_1 < p_2 < N^{\kappa_2} \leq p_3 < N^{1/2-2\kappa_1}/p_2 \\ (p_1 p_2 p_3, N)=1}} \{S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_1) - S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_2)\} \\ & + \sum_{\substack{N^{\kappa_2} \leq p_1 < N^{\sigma_2} \leq p_2 < p_3 < N^{\sigma_1} \\ (p_1 p_2 p_3, N)=1}} \{S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_2) - S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_3)\} \\ = & -\Upsilon_{13} - \Upsilon_{14} + \Upsilon_{15} + O(N^{1-\kappa_1}). \end{aligned}$$

Combining this with (9.9), we obtain the required result. This completes the proof.  $\square$

**Remark 3.** Lemmas 9.1 and 9.2 are also valid for

$$\begin{aligned}\mathcal{A}' &:= \{p+2 : p \leq x\}, \\ \mathcal{A}'' &:= \{p+2 : x < p \leq x+x^\theta\}, \\ \mathcal{A}''' &:= \{N-p : \alpha N < p \leq \alpha N + N^\theta\},\end{aligned}$$

if we make some suitable modifications. For example, we have

$$(9.10) \quad 4\pi_{1,2}(x) \geq 4S(\mathcal{A}'; \mathcal{P}(2), x^{\kappa_1}) - \Upsilon'_1 - \Upsilon'_2 - \Upsilon'_3 + \Upsilon'_4 + \Upsilon'_5 + \Upsilon'_6 - 2\Upsilon'_7 \\ - 2\Upsilon'_8 - \Upsilon'_9 - \Upsilon'_{10} + \Upsilon'_{11} + \Upsilon'_{12} - \Upsilon'_{13} - \Upsilon'_{14} + \Upsilon'_{15} + O(N^{1-\kappa_1}),$$

where  $\Upsilon'_j$  is similarly defined as  $\Upsilon_j$  with the difference that  $\mathcal{A}$  is replaced by  $\mathcal{A}'$ ,  $\mathcal{P}(N)$  by  $\mathcal{P}(2)$ ,  $\mathcal{P}(Np_1)$  by  $\mathcal{P}(2p_1)$ ,  $(N/p_1)^{1/2}$  by  $(x/p_1)^{1/2}$ ,  $N^{1/2-2\kappa_1}/p_2$  by  $x^{4/7-2\kappa_1}/p_2$  (in  $\Upsilon_5$  and  $\Upsilon_{14}$ ),  $N^\rho$  by  $x^\rho$ ,  $N^{\kappa_i}$  by  $x^{\kappa_i}$ ,  $N^{\sigma_i}$  by  $x^{\sigma_i}$  and that the conditions  $(p, N) = 1$ ,  $(p_1p_2, N) = 1$ ,  $(p_1p_2p_3, N) = 1$  and  $(p_1p_2p_3p_4, N) = 1$  are eliminated. The assumption on the parameters is

$$0 < \kappa_1 < \kappa_2 < \rho < \sigma_2 < \sigma_1 < \frac{1}{3}, \quad 3\kappa_1 + \rho \geq \frac{4}{7}.$$

The last condition is necessary in the proof of  $\Delta \geq \Upsilon'_5 + \Delta_1$ .

## § 10. Proofs of Theorems 2 and 5

For simplicity, we write  $\mathcal{L} := \log N$  and use  $B$  to denote a suitable positive constant determined by Lemma 2.3. We shall estimate all terms  $\Upsilon_i$  in the inequality (9.4). For this we suppose that

$$(10.1) \quad \frac{1}{12} = \kappa_1 < \kappa_2 \leq \frac{1}{8}, \quad \frac{1}{4} = \rho < \sigma_2 < \sigma_1 < \frac{1}{3}, \quad 3\sigma_1 + \kappa_1 > 1, \quad 2\sigma_1 + \sigma_2 + \kappa_2 > 1.$$

1° *Lower bound of  $S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_1})$*

We apply (2.5) of Lemma 2.2 with

$$X = \text{li}(N), \quad w(p) = \begin{cases} p/\varphi(p) & \text{if } p \in \mathcal{P}(N), \\ 0 & \text{otherwise,} \end{cases} \quad z = N^{\kappa_1}, \quad Q = \frac{\sqrt{N}}{\mathcal{L}^B}.$$

Since  $|\lambda_l^\pm(q)| \leq 1$ , Lemma 2.3 with the choice  $f(1) = 1$  and  $f(m) = 0$  if  $m \geq 2$  implies that

$$\left| \sum_{l < L} \sum_{q|P(z)} \lambda_l^\pm(q) r(\mathcal{A}, q) \right| \ll_\varepsilon \sum_{q \leq \sqrt{N}/\mathcal{L}^B} \mu(q)^2 \max_{y \leq N} \max_{(a,q)=1} \left| \pi(y; q, a) - \frac{\text{li}(y)}{\varphi(q)} \right| \ll_\varepsilon \frac{N}{\mathcal{L}^3}.$$

In view of  $V(z) \sim 2e^{-\gamma}C_N/\log z$  ( $\gamma$  is the Euler constant) and  $C_N \gg 1$ , we can deduce

$$(10.2) \quad S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_1}) \geq \{F_0 + O(\varepsilon)\} \Theta(N) \quad \text{with} \quad F_0 := 2f(1/2\kappa_1)/\kappa_1 e^\gamma.$$

2° *Upper bounds of  $\Upsilon_1$ ,  $\Upsilon_2$  and  $\Upsilon_3$*

We apply (2.4) of Lemma 2.2 with

$$X = \frac{\text{li}(N)}{\varphi(p)}, \quad w(p) = \begin{cases} p/\varphi(p) & \text{if } p \in \mathcal{P}(N), \\ 0 & \text{otherwise,} \end{cases} \quad z = p, \quad Q = \frac{\sqrt{N}}{p\mathcal{L}^B}$$

to  $S(\mathcal{A}_p; \mathcal{P}(N), p)$ . The contribution of the error term in (2.4) is

$$\begin{aligned} & \ll_{\varepsilon} \sum_{N^{\kappa_1} \leq p < N^{\kappa_2}, (p, N)=1} \sum_{q \leq \sqrt{N}/p\mathcal{L}^B, q|P(p)} |r(\mathcal{A}, pq)| \\ & \ll_{\varepsilon} \sum_{d \leq \sqrt{N}/\mathcal{L}^B} \mu(d)^2 \max_{y \leq N} \max_{(a, d)=1} \left| \pi(y; d, a) - \frac{\text{li}(y)}{\varphi(d)} \right| \ll_{\varepsilon} \frac{N}{\mathcal{L}^3} \end{aligned}$$

by Lemma 2.3 with the same choice of  $f$  as above. Thus

$$\Upsilon_1 \leq \frac{\{1 + O(\varepsilon)\}N}{\mathcal{L}} \sum_{N^{\kappa_1} \leq p < N^{\kappa_2}} \frac{V(p)}{\varphi(p)} F\left(\frac{\log(\sqrt{N}/p)}{\log p}\right) + O\left(\frac{N}{\mathcal{L}^3}\right).$$

The standard procedure for replacing sums over primes by integrals yields

$$(10.3) \quad \Upsilon_1 \leq \{F_1 + O(\varepsilon)\} \Theta(N),$$

where

$$F_1 := \frac{2}{e^{\gamma}} \int_{\kappa_1}^{\kappa_2} \frac{F(1/2t - 1)}{t^2} dt = \frac{4}{e^{\gamma}} \int_{1/2\kappa_2 - 1}^{1/2\kappa_1 - 1} F(t) dt.$$

Similarly we can prove

$$(10.4) \quad \Upsilon_i \leq \{F_i + O(\varepsilon)\} \Theta(N) \quad (i = 2, 3),$$

where

$$F_2 := \frac{4}{e^{\gamma}} \int_{(1/2 - \sigma_1)/\kappa_1}^{(1/2 - \kappa_1)/\kappa_1} \frac{F(t)}{1 - 2\kappa_1 t} dt, \quad F_3 := \frac{4}{e^{\gamma}} \int_{(1/2 - \sigma_2)/\kappa_1}^{(1/2 - \kappa_1)/\kappa_1} \frac{F(t)}{1 - 2\kappa_1 t} dt.$$

3° *Lower bounds of  $\Upsilon_4$  and  $\Upsilon_5$*

As before we can deduce, from Lemmas 2.2 and 2.3, that

$$(10.5) \quad \Upsilon_i \geq \{F_i + O(\varepsilon)\} \Theta(N) \quad (i = 4, 5),$$

where

$$\begin{aligned} F_4 &:= \frac{2}{\kappa_1 e^{\gamma}} \int_{\kappa_1}^{\kappa_2} \frac{dt}{t} \int_t^{\kappa_2} f\left(\frac{1/2 - t - u}{\kappa_1}\right) \frac{du}{u}, \\ F_5 &:= \frac{2}{\kappa_1 e^{\gamma}} \int_{\kappa_1}^{\kappa_2} \frac{dt}{t} \int_{\kappa_2}^{1/2 - 2\kappa_1 - t} f\left(\frac{1/2 - t - u}{\kappa_1}\right) \frac{du}{u}. \end{aligned}$$

We have used the following fact to remove the condition  $(p_1 p_2, N) = 1$ :

$$\sum_{\substack{p_1 | N \\ p_1 \geq N^{\kappa}}} \sum_{N^{\kappa} \leq p_2 < N^{\sigma}} \frac{N}{p_1 p_2} + \sum_{N^{\kappa} \leq p_1 < N^{\sigma}} \sum_{\substack{p_2 | N \\ p_2 \geq N^{\kappa}}} \frac{N}{p_1 p_2} \ll N^{1-\kappa} \mathcal{L}.$$

4° *Upper bounds of  $\Upsilon_i$  for  $i = 7, 8, 9, 10, 13, 14$*

We shall only majorize  $\Upsilon_7$  and the others can be treated similarly.

Since  $\sigma_1 > \frac{1}{4}$ , the quantity  $\Upsilon_7$  is equal to the number of primes  $p \leq N$  such that  $N - p = p_1 p_2 p_3$  with  $N^{\sigma_1} \leq p_1 < p_2 < (N/p_1)^{1/2}$ ,  $p_3 \geq p_2$  and  $(p_1 p_2 p_3, N) = 1$ . Define

$$\begin{aligned} \mathcal{M} &:= \{m : m = p_1 p_2, N^{\sigma_1} \leq p_1 < p_2 < (N/p_1)^{1/2}, (p_1 p_2, N) = 1\}, \\ \mathcal{B} &:= \{b : b = N - mp \leq N, m \in \mathcal{M}, p \leq N/m\}. \end{aligned}$$



It is clear that

$$\Upsilon_7 \leq S(\mathcal{B}; \mathcal{P}(N), N^{1/2}) + O(N^{1/2}).$$

By applying (2.4) of Lemma 2.2 with

$$X = \sum_{m \in \mathcal{M}} \text{li}\left(\frac{N}{m}\right), \quad w(p) = \begin{cases} p/\varphi(p) & \text{if } p \in \mathcal{P}(N), \\ 0 & \text{otherwise,} \end{cases} \quad Q = \frac{\sqrt{N}}{\mathcal{L}^B},$$

we obtain

$$(10.6) \quad \Upsilon_7 \leq \frac{8C_N X}{\log N} \{1 + O(\varepsilon)\} + O_\varepsilon(\sqrt{N} + R_3 + R_4),$$

where

$$R_3 := \sum_{\substack{q \leq \sqrt{N}/\mathcal{L}^B \\ (q, N)=1}} \mu(q)^2 \left| \sum_{\substack{m \in \mathcal{M} \\ (q, m)=1}} \left( \sum_{\substack{mp \leq N \\ mp \equiv N \pmod{q}}} 1 - \frac{\text{li}(N/m)}{\varphi(q)} \right) \right|,$$

$$R_4 := \sum_{q \leq \sqrt{N}/\mathcal{L}^B, (q, N)=1} \frac{\mu(q)^2}{\varphi(q)} \sum_{m \in \mathcal{M}, (q, m) > 1} \text{li}\left(\frac{N}{m}\right).$$

Let  $f(m)$  be the characteristic function of  $\mathcal{M}$ . Since  $m \leq N^{3/4}$  for  $m \in \mathcal{M}$ , Lemma 2.3 implies

$$(10.7) \quad R_3 = \sum_{\substack{q \leq \sqrt{N}/\mathcal{L}^B \\ (q, N)=1}} \mu(q)^2 \left| \sum_{\substack{m \leq N^{5/6} \\ (q, m)=1}} f(m) \left( \sum_{\substack{mp \leq N \\ mp \equiv N \pmod{q}}} 1 - \frac{\text{li}(N/m)}{\varphi(q)} \right) \right| \ll \frac{N}{\mathcal{L}^3}.$$

Noticing that  $(d, m) > 1$  implies  $(d, m) \geq N^{\sigma_1}$  for  $m \in \mathcal{M}$ , we have

$$\begin{aligned} R_4 &\ll \frac{N}{\mathcal{L}} \sum_{q \leq \sqrt{N}} \frac{\mu(q)^2}{\varphi(q)} \sum_{m \leq N^{3/4}, (q, m) \geq N^{\sigma_1}} \frac{1}{m} \\ &\ll \frac{N}{\mathcal{L}} \sum_{q \leq N} \frac{\mu(q)^2}{\varphi(q)} \sum_{d|q, d \geq N^{\sigma_1}} \frac{1}{d} \sum_{n \leq N^{3/4}/d} \frac{1}{n} \\ &\ll N \sum_{q \leq \sqrt{N}} \frac{\mu(q)^2}{\varphi(q)} \sum_{d|q, d \geq N^{\sigma_1}} \frac{1}{d} \\ &\ll N \sum_{N^{\sigma_1} < d \leq N} \frac{1}{d} \sum_{l \leq \sqrt{N}/d} \frac{\mu(dl)^2}{\varphi(dl)} \\ &\ll N \sum_{N^{\sigma_1} < d \leq N} \frac{\mu(d)^2}{d\varphi(d)} \sum_{l \leq \sqrt{N}/d} \frac{\mu(l)^2}{\varphi(l)}. \end{aligned}$$

Since the function  $\mu(n)^2/\varphi(n)$  is multiplicative and  $\mu(p^\nu)^2/\varphi(p^\nu) = 1/(p-1)$  for  $\nu = 1$  and  $= 0$  for  $\nu \geq 2$ , it is plain to see that

$$\sum_{l \leq t} \frac{\mu(l)^2}{\varphi(l)} \asymp \log t.$$

Thus

$$(10.8) \quad R_4 \ll N^{1-\sigma_1} \mathcal{L}^2.$$

By the prime number theorem, we obtain

$$\begin{aligned}
 (10.9) \quad X &= \{1 + o(1)\} \sum_{N^{\sigma_1} \leq p_1 < p_2 < (N/p_1)^{1/2}} \frac{N}{p_1 p_2 \log(N/p_1 p_2)} \\
 &= \{1 + o(1)\} \frac{N}{\mathcal{L}} \int_{\sigma_1}^{1/3} \frac{\log(1/t - 2)}{t(1-t)} dt \\
 &= \{1 + o(1)\} \frac{N}{\mathcal{L}} \int_2^{1/\sigma_1 - 1} \frac{\log(t-1)}{t} dt.
 \end{aligned}$$

Inserting (10.7)–(10.9) into (10.6) yields

$$(10.10) \quad \Upsilon_7 \leq \{F_7 + O(\varepsilon)\} \Theta(N),$$

where

$$F_7 := 8 \int_2^{1/\sigma_1 - 1} \frac{\log(t-1)}{t} dt.$$

Similarly we can prove that

$$(10.11) \quad \Upsilon_i \leq \{F_i + O(\varepsilon)\} \Theta(N) \quad (i = 8, 10),$$

where

$$\begin{aligned}
 F_8 &:= 8 \int_2^{1/\sigma_2 - 1} \frac{\log(t-1)}{t} dt, \\
 F_{10} &:= 8 \int_{\kappa_2}^{\sigma_2} \frac{\log(1/\sigma_2 - 1 - t/\sigma_2)}{t(1-t)} dt.
 \end{aligned}$$

[We need to use the assumption  $2\sigma_1 + \sigma_2 + \kappa_2 > 1$  in  $\Upsilon_{10}$ .]

For the terms  $\Upsilon_9, \Upsilon_{13}$  and  $\Upsilon_{14}$  with  $p_1 \leq N^{1/10}$ , we can apply Lemma 2.6 instead of Lemma 2.3. A similar argument allows us to show that

$$(10.12) \quad \Upsilon_i \leq \{F_i + O(\varepsilon)\} \Theta(N) \quad (i = 9, 13, 14),$$

where

$$\begin{aligned}
 F_9 &:= \frac{36}{5} \int_{\kappa_1}^{1/10} \frac{\log(1/\sigma_1 - 1 - t/\sigma_1)}{t(1-t)^2} dt + 8 \int_{1/10}^{\sigma_1} \frac{\log(1/\sigma_1 - 1 - t/\sigma_1)}{t(1-t)} dt, \\
 F_{13} &:= \frac{36}{5} \int_{\kappa_1}^{1/10} \frac{dt_1}{t_1(1-t_1)} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{t_3}^{\kappa_2} \omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right) \frac{dt_4}{t_4} \\
 &\quad + 8 \int_{1/10}^{\kappa_2} \frac{dt_1}{t_1} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{t_3}^{\kappa_2} \omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right) \frac{dt_4}{t_4}, \\
 F_{14} &:= \frac{36}{5} \int_{\kappa_1}^{1/10} \frac{dt_1}{t_1(1-t_1)} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{\kappa_2}^{1/2-2\kappa_1-t_3} \omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right) \frac{dt_4}{t_4} \\
 &\quad + 8 \int_{1/10}^{\kappa_2} \frac{dt_1}{t_1} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{\kappa_2}^{1/2-2\kappa_1-t_3} \omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right) \frac{dt_4}{t_4}.
 \end{aligned}$$

[We need to use the assumption  $3\sigma_1 + \kappa_1 > 1$  in  $\Upsilon_9$  and Lemma 2.10 in  $\Upsilon_{13}$  and  $\Upsilon_{14}$ .]

By inserting (10.2)–(10.5), (10.10)–(10.12) and by using the trivial lower bounds  $\Upsilon_i \geq 0$  ( $i = 6, 11, 12, 15$ ) into (9.4), we get the following inequality

$$D_{1,2}(N) \geq \{F(\kappa_1, \kappa_2, \rho, \sigma_2, \sigma_1) + O(\varepsilon)\} \Theta(N),$$

where

$$F(\kappa_1, \kappa_2, \rho, \sigma_2, \sigma_1) := \frac{1}{4}(4F_0 - F_1 - F_2 - F_3 + F_4 + F_5 \\ - 2F_7 - 2F_8 - F_9 - F_{10} - F_{13} - F_{14}).$$

Taking  $\kappa_1 = \frac{1}{12}$ ,  $\kappa_2 = \frac{29}{250}$ ,  $\rho = \frac{1}{4}$ ,  $\sigma_2 = \frac{141}{500}$  and  $\sigma_1 = \frac{41}{125}$ , a numerical computation gives us

$$F(\kappa_1, \kappa_2, \rho, \sigma_2, \sigma_1) \geq \frac{1}{4}(4 \times 13.473613 - 3.891854 - 20.432098 - 17.327241 \\ + 0.697375 + 2.118119 - 2 \times 0.004609 - 2 \times 0.434368 \\ - 5.161945 - 5.468377 - 0.023310 - 0.182860) \\ > 0.83607.$$

[For the integrals  $F_{13}$  and  $F_{14}$ , we make use of  $\omega(u) \leq 0.561522$  for  $u \geq 3.5$ .] This completes the proof of Theorem 2.  $\square$

Theorem 5 can be proved in the same way. The only difference is to replace Lemmas 2.3 and 2.6 by Lemma 2.4. Here, the choice of parameters is

$$(\theta, \kappa_1, \kappa_2, \rho, \sigma_2, \sigma_1) = (0.971, (2\theta - 1)/12, 0.111, (2\theta - 1)/4, 0.271, 0.313). \quad \square$$

## § 11. Proof of Theorem 4

The proof of Theorem 4 is very similar to that of Theorem 2. But we must use Lemmas 2.5, 2.7 and 2.8 in place of Lemmas 2.3 and 2.6. In order to take the advantage of these lemmas, we must carry out a more careful and delicate analysis. Thus the proof will be slightly complicated.

Suppose that the parameters satisfy the following conditions:

$$\frac{2}{21} = \kappa_1 < \kappa_2 \leq \frac{1}{7}, \quad \frac{2}{7} = \rho < \sigma_2 < \frac{29}{100} < \sigma_1 < \frac{1}{3}, \quad 3\sigma_1 + \kappa_1 > 1, \quad 2\sigma_1 + \sigma_2 + \kappa_2 > 1.$$

1° *Lower bounds of  $S(\mathcal{A}'; \mathcal{P}(2), x^{\kappa_1})$*

By (2.5) of Lemma 2.2 and Lemma 2.7, we can easily prove

$$(11.1) \quad S(\mathcal{A}'; \mathcal{P}(2), x^{\kappa_1}) \geq \{G_0 + O(\varepsilon)\} \Pi(x) \quad \text{with} \quad G_0 := f(4/7\kappa_1)/\kappa_1 e^\gamma.$$

2° *Upper bounds of  $\Upsilon'_1$ ,  $\Upsilon'_2$  and  $\Upsilon'_3$*

We divide the interval  $[x^{\kappa_1}, x^{\kappa_2}]$  into  $O(\mathcal{L})$  subintervals of the form  $[P, 2P]$  and apply (2.4) of Lemma 2.2 to  $S(\mathcal{A}'_p; \mathcal{P}(2), p)$  for  $p \in [P, 2P]$ . We have

$$S(\mathcal{A}'_p; \mathcal{P}(2), p) \leq \frac{\{1 + O(\varepsilon)\}x}{\mathcal{L}} \frac{V(p)}{\varphi(p)} F\left(\frac{\log(Q/P)}{\log p}\right) + \sum_{l < L} \sum_{q|P(p)} \lambda_l^+(q) r(\mathcal{A}', pq)$$

where  $Q = x^{4/7-\varepsilon}$  and  $\lambda_l^+(q)$  is well factorable of level  $Q/P$  and of order 1.

Denote by  $\pi_P$  the characteristic function of the primes in the interval  $[P, 2P]$ . Noticing that  $P \leq x^{\kappa_2} \Rightarrow P \leq Q/P$ , Lemma 2.1 shows that  $\pi_P * \lambda_l^+$  is well factorable of level  $Q$  and of order 2. Thus Lemma 2.7 allows us to deduce that

$$\sum_{P \leq p < 2P} \sum_{l < L} \sum_{q|P(p)} \lambda_l^+(q) r(\mathcal{A}', pq) \ll_\varepsilon x/(\log x)^4$$

and

$$\Upsilon'_1 \leq \frac{\{1 + O(\varepsilon)\}x}{\log x} \sum_{x^{\kappa_1} \leq p < x^{\kappa_2}} \frac{V(p)}{\varphi(p)} F\left(\frac{\log(Q/p)}{\log p}\right) + O\left(\frac{x}{(\log x)^4}\right).$$

Since  $V(p) \sim e^{-\gamma} C / \log p$ , the prime number theorem implies that

$$(11.2) \quad \Upsilon'_1 \leq \{G_1 + O(\varepsilon)\} \Pi(x) \quad \text{with} \quad G_1 := \frac{7}{4e^\gamma} \int_{4/7\kappa_2-1}^{4/7\kappa_1-1} F(t) dt.$$

We divide the interval of summation  $[x^{\kappa_1}, x^{\sigma_1}]$  of  $\Upsilon'_2$  into three parts:

$$[x^{\kappa_1}, x^{2/7-\varepsilon}], \quad [x^{2/7-\varepsilon}, x^{29/100}], \quad [x^{29/100}, x^{\sigma_1}],$$

and use (2.4) of Lemma 2.2 to handle each sum. As before we apply Lemma 2.7, the condition (C.2) and (C.3) of Lemma 2.8, respectively, to control the corresponding error terms. We find

$$(11.3) \quad \Upsilon'_2 \leq \{G_2 + O(\varepsilon)\} \Pi(x),$$

where

$$G_2 := \frac{1}{\kappa_1 e^\gamma} \left\{ \int_{2/7\kappa_1}^{4/7\kappa_1-1} \frac{F(t)}{4/7\kappa_1-t} dt + \int_{13/50\kappa_1}^{2/7\kappa_1} \frac{F(t)}{2/\kappa_1-t} dt + \int_{(11/20-\sigma_1)/\kappa_1}^{13/50\kappa_1} \frac{F(t)}{11/20\kappa_1-t} dt \right\}.$$

Similarly

$$\Upsilon'_3 \leq \{G_3 + O(\varepsilon)\} \Pi(x),$$

where

$$(11.4) \quad G_3 := \frac{1}{\kappa_1 e^\gamma} \left\{ \int_{2/7\kappa_1}^{(4/7-\kappa_1)/\kappa_1} \frac{F(t)}{4/7\kappa_1-t} dt + \int_{(2-6\sigma_2)/\kappa_1}^{2/7\kappa_1} \frac{F(t)}{2/\kappa_1-t} dt \right\}.$$

3° Lower bounds of  $\Upsilon'_4$  and  $\Upsilon'_5$

In view of  $3\kappa_2 \leq \frac{3}{7} < \frac{4}{7}$ , a similar argument proving (11.2) implies that

$$(11.5) \quad \Upsilon'_4 \geq \{G_4 + O(\varepsilon)\} \Pi(x),$$

where

$$G_4 := \frac{1}{\kappa_1 e^\gamma} \int_{\kappa_1}^{\kappa_2} \frac{dt}{t} \int_t^{\kappa_2} f\left(\frac{4/7-t-u}{\kappa_1}\right) \frac{du}{u}.$$

Our assumptions on  $\kappa_1$  and  $\rho$  imply that  $p_1^2 p_2 \leq x^{4/7-\varepsilon}$  and  $p_1^2 \leq x^{4/7-\varepsilon}$ . As before we can apply (2.5) of Lemma 2.2 and Lemma 2.7 to get

$$(11.6) \quad \Upsilon'_5 \geq \{G_5 + O(\varepsilon)\} \Pi(x),$$

where

$$G_5 := \frac{1}{\kappa_1 e^\gamma} \int_{\kappa_1}^{\kappa_2} \frac{dt}{t} \int_{\kappa_2}^{4/7-2\kappa_1-t} f\left(\frac{4/7-t-u}{\kappa_1}\right) \frac{du}{u}.$$

4° Upper bounds of  $\Upsilon'_i$  for  $i = 7, 8, 9, 10, 13, 14$

We shall apply the technique of [12]. Since  $\sigma_1 > \frac{2}{7}$ , the quantity  $\Upsilon'_7$  is equal to the number of primes  $p \leq x$  such that  $p+2 = p_1 p_2 p_3$  with  $x^{\sigma_1} \leq p_1 < p_2 < (x/p_1)^{1/2}$  and  $p_3 \geq p_2$ .

Introduce the set

$$\mathcal{B} := \{b - 2 : b = p_1 p_2 p_3 \leq x, x^{\sigma_1} \leq p_1 < p_2 < p_3\}.$$

Then we have

$$\Upsilon'_7 = S(\mathcal{B}; \mathcal{P}(2), x^{1/2}) + O(x^{1/2}).$$

Let  $\Delta := 1 + \mathcal{L}^{-4}$ . We cover the set  $\mathcal{B}$  by cuboids

$$\mathcal{B}(t_1, t_2, t_3) := \{b - 2 : b = p_1 p_2 p_3 \leq x, p_i \in [\Delta^{t_i}, \Delta^{t_i+1}) \text{ for } 1 \leq i \leq 3\}$$

where  $t_i$  are integers satisfying  $x^{\sigma_1} \leq \Delta^{t_1} \leq \Delta^{t_2} \leq \Delta^{t_3}$  and  $\Delta^{t_1+t_2+t_3+3} \leq x$ . In view of  $x^{2/7} \leq p_2 \leq x^{(1-\sigma_2)/2} \leq x^{2/5}$ , Lemma 2.5 with the choice

$$\alpha_m = \begin{cases} 1 & \text{if } m = p_1 p_3 \\ 0 & \text{otherwise} \end{cases}, \quad \beta_n = \begin{cases} 1 & \text{if } n = p_2 \\ 0 & \text{otherwise} \end{cases}$$

implies the inequality

$$\sum_{(q,2)=1} \lambda_l^+(q) \left( |\mathcal{B}(t_1, t_2, t_3)_q| - \frac{|\mathcal{B}(t_1, t_2, t_3)|}{\varphi(q)} \right) \ll \frac{x}{(\log x)^{18}},$$

where  $\lambda_l^+(q)$  is well factorable of order 1 and of level  $Q = x^{\theta(t_2)}$  with  $\theta(t_2) = (2 + t_2)/4$ .

Thus we find by (2.4) of Lemma 2.2,

$$S(\mathcal{B}(t_1, t_2, t_3); \mathcal{P}(2), x^{1/2}) \leq \frac{2C\{1 + O(\varepsilon)\}}{\theta(t_2)\mathcal{L}} |\mathcal{B}(t_1, t_2, t_3)| + O\left(\frac{x}{(\log x)^{18}}\right).$$

Since the number of cuboids  $\mathcal{B}(t_1, t_2, t_3)$  is  $O((\log x)^{15})$ , we have

$$\begin{aligned} \sum_{(t_1, t_2, t_3)} \frac{|\mathcal{B}(t_1, t_2, t_3)|}{\theta(t_2)} &= \sum_{x^{\sigma_1} \leq p_1 < p_2 \leq (x/p_1)^{1/2}} \sum \frac{4x}{p_1 p_2 \log(x/p_1 p_2)(2 + \log p_2 / \log x)} + O\left(\frac{x}{(\log x)^2}\right) \\ &= \frac{4x\{1 + O(\varepsilon)\}}{\log x} \int \int_{\sigma_1 \leq t \leq u \leq (1-t)/2} \frac{dt du}{tu(1-t-u)(2+u)}. \end{aligned}$$

Combining these estimates, we obtain

$$(11.7) \quad \Upsilon'_7 \leq \{G_7 + O(\varepsilon)\} \Pi(x),$$

where

$$G_7 := 8 \int \int_{\sigma_1 \leq t \leq u \leq (1-t)/2} \frac{dt du}{tu(1-t-u)(2+u)}.$$

Analogously we have

$$(11.8) \quad \Upsilon'_8 \leq \{G_8 + O(\varepsilon)\} \Pi(x),$$

where

$$G_8 := 8 \int \int_{\sigma_2 \leq t \leq u \leq (1-t)/2} \frac{dt du}{tu(1-t-u)(2+u)}.$$

For  $\Upsilon'_9$ , the assumption  $3\sigma_1 + \kappa_1 > 1$  allows us to write

$$\Upsilon'_9 = S(\mathcal{B}'; \mathcal{P}(2), x^{1/2}) + O(x^{1/2})$$

with

$$(11.9) \quad \mathcal{B}' := \{b - 2 : b = p_1 p_2 p_3 \leq x, x^{\kappa_1} \leq p_1 < x^{\sigma_1} \leq p_2 < p_3\}.$$

We decompose  $\mathcal{B}' = \mathcal{B}'_1 \cup \dots \cup \mathcal{B}'_6$ , where  $\mathcal{B}'_1, \dots, \mathcal{B}'_6$  are defined as in (11.9) but we add respectively the extra conditions

$$\begin{aligned} p_1 &\leq x^{1/10} \text{ in } \mathcal{B}'_1; \\ p_1 &> x^{1/10} \text{ and } p_1 p_2 \leq x^{1/2} \text{ in } \mathcal{B}'_2; \\ p_1 &> x^{1/10} \text{ and } p_1^{-2} p_2^8 > x^3 \text{ in } \mathcal{B}'_3; \\ p_2 &> x^{2/5} \text{ and } p_1^{-2} p_2^8 \leq x^3 \text{ in } \mathcal{B}'_4; \\ x^{(1-\sigma_2)/2} &< p_2 \leq x^{2/5} \text{ and } p_1 p_2 > x^{1/2} \text{ in } \mathcal{B}'_5; \\ p_2 &\leq x^{(1-\sigma_2)/2} \text{ and } p_1 p_2 > x^{1/2} \text{ in } \mathcal{B}'_6. \end{aligned}$$

Again we can use Lemma 2.5 with

$$\begin{aligned} \theta(\log p_1 / \log x) &= (1 + 2 \log p_1 / \log x) / 2 \text{ for } \mathcal{B}'_1; \\ \theta(\log p_1 / \log x) &= (5 - 2 \log p_1 / \log x) / 8 \text{ for } \mathcal{B}'_2 \text{ and } \mathcal{B}'_3; \\ \theta(\log p_2 / \log x) &= 1 - \log p_2 / \log x \text{ for } \mathcal{B}'_4; \\ \theta(\log p_2 / \log x) &= (2 + \log p_2 / \log x) / 4 \text{ for } \mathcal{B}'_5 \text{ and } \mathcal{B}'_6. \end{aligned}$$

Then we have

$$(11.10) \quad \Upsilon'_9 \leq \{G_9 + O(\varepsilon)\} \Pi(x),$$

where

$$\begin{aligned} G_9 &= 4 \int \int_{\substack{\kappa_1 \leq t \leq \sigma_1 \leq u \leq (1-t)/2 \\ t \leq 1/10}} \frac{dt du}{tu(1-t-u)(1+2t)} \\ &+ 16 \int \int_{\substack{\kappa_1 \leq t \leq \sigma_1 \leq u \leq (1-t)/2 \\ t \geq 1/10, t+u \leq 1/2}} \frac{dt du}{tu(1-t-u)(5-2t)} \\ &+ 16 \int \int_{\substack{\kappa_1 \leq t \leq \sigma_1 \leq u \leq (1-t)/2 \\ t \geq 1/10, 8u \geq 2t+3}} \frac{dt du}{tu(1-t-u)(5-2t)} \\ &+ 2 \int \int_{\substack{\kappa_1 \leq t \leq \sigma_1 \leq u \leq (1-t)/2 \\ u \geq 2/5, 8u \leq 2t+3}} \frac{dt du}{tu(1-t-u)(1-u)} \\ &+ 8 \int \int_{\substack{\kappa_1 \leq t \leq \sigma_1 \leq u \leq (1-t)/2 \\ (1-\sigma_1)/2 \leq u \leq 2/5, t+u \geq 1/2}} \frac{dt du}{tu(1-t-u)(2+u)} \\ &+ 8 \int \int_{\substack{\kappa_1 \leq t \leq \sigma_1 \leq u \leq (1-t)/2 \\ u \leq (1-\sigma_1)/2, t+u \geq 1/2}} \frac{dt du}{tu(1-t-u)(2+u)}. \end{aligned}$$

Similarly in view of the assumption  $2\sigma_1 + \sigma_2 + \kappa_2 > 1$ , we can prove

$$(11.11) \quad \Upsilon'_{10} \leq \{G_{10} + O(\varepsilon)\} \Pi(x),$$

where

$$\begin{aligned} G_{10} = & 16 \int_{\substack{\kappa_2 \leq t \leq \sigma_2 \leq u \leq (1-t)/2 \\ t+u \leq 1/2}} \int \frac{dt du}{tu(1-t-u)(5-2t)} \\ & + 16 \int_{\substack{\kappa_2 \leq t \leq \sigma_2 \leq u \leq (1-t)/2 \\ 8u \geq 2t+3}} \int \frac{dt du}{tu(1-t-u)(5-2t)} \\ & + 2 \int_{\substack{\kappa_2 \leq t \leq \sigma_2 \leq u \leq (1-t)/2 \\ 2/5 \leq u \leq (2\kappa_2+3)/8}} \int \frac{dt du}{tu(1-t-u)(1-u)} \\ & + 2 \int_{\substack{\kappa_2 \leq t \leq \sigma_2 \leq u \leq (1-t)/2 \\ u \geq (2\kappa_2+3)/8, 8u \leq 2t+3}} \int \frac{dt du}{tu(1-t-u)(1-u)} \\ & + 8 \int_{\substack{\kappa_2 \leq t \leq \sigma_2 \leq u \leq (1-t)/2 \\ 1/2 - \kappa_2 \leq u \leq 2/5, t+u \geq 1/2}} \int \frac{dt du}{tu(1-t-u)(2+u)} \\ & + 8 \int_{\substack{\kappa_2 \leq t \leq \sigma_2 \leq u \leq (1-t)/2 \\ (1-\sigma_2)/2 \leq u \leq 1/2 - \kappa_2, t+u \geq 1/2}} \int \frac{dt du}{tu(1-t-u)(2+u)} \\ & + 8 \int_{\substack{\kappa_2 \leq t \leq \sigma_2 \leq u \leq (1-t)/2 \\ u \leq (1-\sigma_2)/2, t+u \geq 1/2}} \int \frac{dt du}{tu(1-t-u)(2+u)}. \end{aligned}$$

More easily we can prove that

$$(11.12) \quad \Upsilon_i \leq \{G_i + O(\varepsilon)\} \Theta(N) \quad (i = 13, 14),$$

where

$$\begin{aligned} G_{13} := & 4 \int_{\kappa_1}^{1/10} \frac{dt_1}{t_1(1+2t_1)} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{t_3}^{\kappa_2} \omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right) \frac{dt_4}{t_4} \\ & + 16 \int_{1/10}^{\kappa_2} \frac{dt_1}{t_1(5-2t_1)} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{t_3}^{\kappa_2} \omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right) \frac{dt_4}{t_4}, \\ G_{14} := & 4 \int_{\kappa_1}^{1/10} \frac{dt_1}{t_1(1+2t_1)} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{\kappa_2}^{4/7-2\kappa_1-t_3} \omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right) \frac{dt_4}{t_4} \\ & + 16 \int_{1/10}^{\kappa_2} \frac{dt_1}{t_1(5-2t_1)} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{\kappa_2}^{4/7-2\kappa_1-t_3} \omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right) \frac{dt_4}{t_4}. \end{aligned}$$

Inserting these estimations and the trivial lower bounds  $\Upsilon'_i \geq 0$  ( $i = 6, 11, 12, 15$ ) into (9.10), we obtain

$$\pi_{1,2}(x) \geq \{G(\kappa_1, \kappa_2, \rho, \sigma_2, \sigma_1) + O(\varepsilon)\} \Pi(x),$$

where

$$\begin{aligned} G(\kappa_1, \kappa_2, \rho, \sigma_2, \sigma_1) := & \frac{1}{4}(4G_0 - G_1 - G_2 - G_3 + G_4 + G_5 \\ & - 2G_7 - 2G_8 - G_9 - G_{10} - G_{13} - G_{14}). \end{aligned}$$

Taking  $\kappa_1 = \frac{2}{21}$ ,  $\kappa_2 = \frac{13}{100}$ ,  $\rho = \frac{2}{7}$ ,  $\sigma_2 = \frac{36}{125}$  and  $\sigma_1 = \frac{332}{1000}$ , a numerical computation gives us

$$\begin{aligned} G(\kappa_1, \kappa_2, \rho, \sigma_2, \sigma_1) &\geq \frac{1}{4}(4 \times 5.894705 - 1.611441 - 7.921437 - 6.736885 \\ &\quad + 0.270916 + 0.913995 - 2 \times 0.000124 - 2 \times 0.145114 \\ &\quad - 1.790090 - 1.930545 - 0.006814 - 0.059690) \\ &> 1.10409. \end{aligned}$$

[For the integrals  $F_{13}$  and  $F_{14}$ , we make use of  $\omega(u) \leq 0.561522$  for  $u \geq 3.5$  and  $\omega(u) \leq 0.567144$  for  $u \geq 2$ .] This completes the proof of Theorem 4.  $\square$

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